

## 1 Shuttles and Taxis at Airport

Note 19

In front of terminal 3 at San Francisco Airport is a pickup area where shuttles and taxis arrive according to a Poisson distribution. The shuttles arrive at a rate  $\lambda_1 = 1/20$  (i.e. 1 shuttle per 20 minutes) and the taxis arrive at a rate  $\lambda_2 = 1/10$  (i.e. 1 taxi per 10 minutes) starting at 00:00. The shuttles and the taxis arrive independently.

- (a) What is the distribution of the following:
- (i) The number of taxis that arrive between times 00:00 and 00:20?
  - (ii) The number of shuttles that arrive between times 00:00 and 00:20?
  - (iii) The total number of pickup vehicles that arrive between times 00:00 and 00:20?
- (b) What is the probability that exactly 1 shuttle and 3 taxis arrive between times 00:00 and 00:20?
- (c) Given that exactly 1 pickup vehicle arrived between times 00:00 and 00:20, what is the conditional probability that this vehicle was a taxi?
- (d) Suppose you reach the pickup area at 00:20. You learn that you missed 3 taxis and 1 shuttle in those 20 minutes. What is the probability that you need to wait for more than 10 mins until either a shuttle or a taxi arrives?

## 2 Family Planning

Note 15

Mr. and Mrs. Johnson decide to continue having children until they either have their first girl or until they have three children. Assume that each child is equally likely to be a boy or a girl, independent of all other children, and that there are no multiple births. Let  $G$  denote the numbers of girls that the Johnsons have. Let  $C$  be the total number of children they have.

(a) Determine the sample space, along with the probability of each sample point.

(b) Compute the joint distribution of  $G$  and  $C$ . Fill in the table below.

	$C = 1$	$C = 2$	$C = 3$
$G = 0$			
$G = 1$			

(c) Use the joint distribution to compute the marginal distributions of  $G$  and  $C$  and confirm that the values are as you'd expect. Fill in the tables below.

$\mathbb{P}[G = 0]$		$\mathbb{P}[C = 1]$	$\mathbb{P}[C = 2]$	$\mathbb{P}[C = 3]$
$\mathbb{P}[G = 1]$				

(d) Are  $G$  and  $C$  independent?

(e) What is the expected number of girls the Johnsons will have? What is the expected number of children that the Johnsons will have?

### 3 Linearity

Note 15

Solve each of the following problems using linearity of expectation. Explain your methods clearly.

- (a) In an arcade, you play game  $A$  10 times and game  $B$  20 times. Each time you play game  $A$ , you win with probability  $1/3$  (independently of the other times), and if you win you get 3 tickets (redeemable for prizes), and if you lose you get 0 tickets. Game  $B$  is similar, but you win with probability  $1/5$ , and if you win you get 4 tickets. What is the expected total number of tickets you receive?
- (b) A monkey types at a 26-letter keyboard with one key corresponding to each of the lower-case English letters. Each keystroke is chosen independently and uniformly at random from the 26 possibilities. If the monkey types 1 million letters, what is the expected number of times the sequence “book” appears? (*Hint*: Consider where the sequence “book” can appear in the string.)

## 4 Balls in Bins

Note 15

You are throwing  $k$  balls into  $n$  bins. Let  $X_i$  be the number of balls thrown into bin  $i$ .

(a) What is  $\mathbb{E}[X_i]$ ?

(b) What is the expected number of empty bins?

(c) Define a collision to occur when a ball lands in a nonempty bin (if there are  $n$  balls in a bin, count that as  $n - 1$  collisions). What is the expected number of collisions?