1 LLSE

Note 20

We have two bags of balls. The fractions of red balls and blue balls in bag A are $2/3$ and $1/3$ respectively. The fractions of red balls and blue balls in bag B are $1/2$ and $1/2$ respectively. Someone gives you one of the bags (unmarked) uniformly at random. You then draw 6 balls from that same bag with replacement. Let X_i be the indicator random variable that ball i is red. Now, let us define $X = \sum_{1 \le i \le 3} X_i$ and $Y = \sum_{4 \le i \le 6} X_i$.

(a) Compute $\mathbb{E}[X]$ and $\mathbb{E}[Y]$.

(b) Compute $\text{Var}(X)$.

(c) Compute $\text{cov}(X,Y)$. (*Hint*: Recall that covariance is bilinear.)

(d) Now, we are going to try and predict Y from a value of X. Compute $L(Y \mid X)$, the best linear estimator of Y given X. Recall that

$$L(Y \mid X) = \mathbb{E}[Y] + \frac{\text{cov}(X,Y)}{\text{Var}(X)}\big(X - \mathbb{E}[X]\big).$$

2 Balls in Bins Estimation

Note 20

We throw $n > 0$ balls into $m \ge 2$ bins. Let X and Y represent the number of balls that land in bin 1 and 2 respectively.

(a) Calculate $\mathbb{E}[Y \mid X]$. [*Hint*: Your intuition may be more useful than formal calculations.]

(b) What is $L[Y \mid X]$ (where $L[Y \mid X]$ is the best linear estimator of Y given X)? [*Hint*: Your justification should be no more than two or three sentences, no calculations necessary! Think carefully about the meaning of the conditional expectation.]

(c) Unfortunately, your friend is not convinced by your answer to the previous part. Compute $\mathbb{E}[X]$ and $\mathbb{E}[Y]$.

(d) Compute $\text{Var}(X)$.

(e) Compute $\text{cov}(X,Y)$.

(f) Compute $L[Y \mid X]$ using the formula. Ensure that your answer is the same as your answer to part (b).

3 Number of Ones

Note 20

In this problem, we will revisit dice-rolling, except with conditional expectation. (*Hint*: for both of these subparts, the law of total expectation may be helpful.)

(a) If we roll a die until we see a 6, how many ones should we expect to see?

(b) If we roll a die until we see a number greater than 3, how many ones should we expect to see?