

Due: Saturday, 10/21, 4:00 PM
Grace period until Saturday, 10/21, 6:00 PM

Sundry

Before you start writing your final homework submission, state briefly how you worked on it. Who else did you work with? List names and email addresses. (In case of homework party, you can just describe the group.)

1 Countability Basics

Note 11

- (a) Is $f : \mathbb{N} \rightarrow \mathbb{N}$, defined by $f(n) = n^2$, an injection (one-to-one)? Briefly justify.
- (b) Is $f : \mathbb{R} \rightarrow \mathbb{R}$, defined by $f(x) = x^3 + 1$, a surjection (onto)? Briefly justify.
- (c) The Bernstein-Schroder theorem states that, if there exist injective functions $f : A \rightarrow B$ and $g : B \rightarrow A$ between the sets A and B , then a bijection exists between A and B . Use this to demonstrate that $(0, 1)$ and $\mathbb{R}_+ = (0, \infty)$ have the same cardinality by defining appropriate injections.

2 Unprogrammable Programs

Note 12

Prove whether the programs described below can exist or not.

- (a) A program $P(F, x, y)$ that returns true if the program F outputs y when given x as input (i.e. $F(x) = y$) and false otherwise.
- (b) A program P that takes two programs F and G as arguments, and returns true if F and G halt on the same set of inputs (or false otherwise).

3 Fixed Points

Note 12

Consider the problem of determining if a program P has any fixed points. Given any program P , a fixed point is an input x such that $P(x)$ outputs x .

- (a) Prove that the problem of determining whether a program has a fixed point is uncomputable.

- (b) Consider the problem of outputting a fixed point of a program if it has one, and outputting "Null" otherwise. Prove that this problem is uncomputable.
- (c) Consider the problem of outputting a fixed point of a program F if the fixed point exists *and* is a natural number, and outputting "Null" otherwise. If an input is a natural number, then it has no leading zero before its most significant bit.

Show that if this problem can be solved, then the problem in part (b) can be solved. What does this say about the computability of this problem? (You may assume that the set of all possible inputs to a program is countable, as is the case on your computer.)

4 Probability Warm-Up

Note 13

- (a) Suppose that we have a bucket of 30 green balls and 70 orange balls. If we pick 15 balls uniformly out of the bucket, what is the probability of getting exactly k green balls (assuming $0 \leq k \leq 15$) if the sampling is done **with** replacement, i.e. after we take a ball out the bucket we return the ball back to the bucket for the next round?
- (b) Same as part (a), but the sampling is **without** replacement, i.e. after we take a ball out the bucket we **do not** return the ball back to the bucket.
- (c) If we roll a regular, 6-sided die 5 times. What is the probability that at least one value is observed more than once?

5 Five Up

Note 13

Say you toss a coin five times, and record the outcomes. For the three questions below, you can assume that order matters in the outcome, and that the probability of heads is some p in $0 < p < 1$, but *not* that the coin is fair ($p = 0.5$).

- (a) What is the size of the sample space, $|\Omega|$?
- (b) How many elements of Ω have exactly three heads?
- (c) How many elements of Ω have three or more heads?

For the next three questions, you can assume that the coin is fair (i.e. heads comes up with $p = 0.5$, and tails otherwise).

- (d) What is the probability that you will observe the sequence HHHTT? What about HHHHT?
- (e) What is the probability of observing at least one head?
- (f) What is the probability you will observe more heads than tails?

For the final three questions, you can instead assume the coin is biased so that it comes up heads with probability $p = \frac{2}{3}$.

- (g) What is the probability of observing the outcome HHHTT? What about HHHHT?
- (h) What about the probability of at least one head?
- (i) What is the probability you will observe more heads than tails?