CS 70	Discrete Mathematics and Probability Theory	
Fall 2023	Tal, Rao	HW 10

Due: Saturday, 11/4, 4:00 PM
Grace period until Saturday, 11/4, 6:00 PM

Sundry

Before you start writing your final homework submission, state briefly how you worked on it. Who else did you work with? List names and email addresses. (In case of homework party, you can just describe the group.)

1 Random Variables Warm-Up

Note 15

Let X and Y be random variables, each taking values in the set $\{0,1,2\}$, with joint distribution

$$\begin{array}{lll}
\mathbb{P}[X=0,Y=0]=1/3 & \mathbb{P}[X=0,Y=1]=0 & \mathbb{P}[X=0,Y=2]=1/3 \\
\mathbb{P}[X=1,Y=0]=0 & \mathbb{P}[X=1,Y=1]=1/9 & \mathbb{P}[X=1,Y=2]=0 \\
\mathbb{P}[X=2,Y=0]=1/9 & \mathbb{P}[X=2,Y=1]=1/9 & \mathbb{P}[X=2,Y=2]=0.
\end{array}$$

(a) What are the marginal distributions of X and Y?

(b) What are $\mathbb{E}[X]$ and $\mathbb{E}[Y]$?

(c) Let I be the indicator that $X=1$, and J be the indicator that $Y=1$. What are $\mathbb{E}[I]$, $\mathbb{E}[J]$ and $\mathbb{E}[IJ]$?

(d) In general, let I_A and I_B be the indicators for events A and B in a probability space $(\Omega,\mathbb{P})$. What is $\mathbb{E}[I_A I_B]$, in terms of the probability of some event?

2 Testing Model Planes

Note 15

Amin is testing model airplanes. He starts with n model planes which each independently have probability p of flying successfully each time they are flown, where $0<p<1$. Each day, he flies every single plane and keeps the ones that fly successfully (i.e. don't crash), throwing away all other models. He repeats this process for many days, where each "day" consists of Amin flying all remaining model planes and throwing away any that crash. Let X_i be the random variable representing how many model planes remain after i days. Note that $X_0=n$. Justify your answers for each part.

(a) What is the distribution of X_1? That is, what is $\mathbb{P}[X_1 = k]$?

(b) What is the distribution of X_2? That is, what is $\mathbb{P}[X_2 = k]$? Recognize the distribution of X_2 as one of the famous ones and provide its name and parameters.

(c) Repeat the previous part for X_t for arbitrary $t \geq 1$.

(d) What is the probability that at least one model plane still remains (has not crashed yet) after t days? Do not have any summations in your answer.

(e) Considering only the first day of flights, is the event A_1 that the first and second model planes crash independent from the event B_1 that the second and third model planes crash? Recall that two events A and B are independent if $\mathbb{P}[A \cap B] = \mathbb{P}[A]\mathbb{P}[B]$. Prove your answer using this definition.

(f) Considering only the first day of flights, let A_2 be the event that the first model plane crashes *and* exactly two model planes crash in total. Let B_2 be the event that the second plane crashes on the first day. What must n be equal to in terms of p such that A_2 is independent from B_2? Prove your answer using the definition of independence stated in the previous part.

(g) Are the random variables X_i and X_j, where $i < j$, independent? Recall that two random variables X and Y are independent if $\mathbb{P}[X = k_1 \cap Y = k_2] = \mathbb{P}[X = k_1]\mathbb{P}[Y = k_2]$ for all k_1 and k_2. Prove your answer using this definition.

3 Class Enrollment

Note 15
Note 19

Lydia has just started her CalCentral enrollment appointment. She needs to register for a geography class and a history class. There are no waitlists, and she can attempt to enroll once per day in either class or both. The CalCentral enrollment system is strange and picky, so the probability of enrolling successfully in the geography class on each attempt is p_g and the probability of enrolling successfully in the history class on each attempt is p_h. Also, these events are independent.

(a) Suppose Lydia begins by attempting to enroll in the geography class everyday and gets enrolled in it on day G. What is the distribution of G?

(b) Suppose she is not enrolled in the geography class after attempting each day for the first 7 days. What is $\mathbb{P}[G = i \mid G > 7]$, the conditional distribution of G given $G > 7$?

(c) Once she is enrolled in the geography class, she starts attempting to enroll in the history class from day $G + 1$ and gets enrolled in it on day H. Find the expected number of days it takes Lydia to enroll in both the classes, i.e. $\mathbb{E}[H]$.

Suppose instead of attempting one by one, Lydia decides to attempt enrolling in both the classes from day 1. Let G be the number of days it takes to enroll in the geography class, and H be the number of days it takes to enroll in the history class.

(d) What is the distribution of G and H now? Are they independent?

(e) Let A denote the day she gets enrolled in her first class and let B denote the day she gets enrolled in both the classes. What is the distribution of A?

(f) What is the expected number of days it takes Lydia to enroll in both classes now, i.e. $\mathbb{E}[B]$?

(g) What is the expected number of classes she will be enrolled in by the end of 30 days?

4 Geometric and Poisson

Note 19

Let $X \sim \text{Geometric}(p)$ and $Y \sim \text{Poisson}(\lambda)$ be independent random variables. Compute $\mathbb{P}[X > Y]$. Your final answer should not have summations.

Hint: Use the total probability rule.

5 Random Tournaments

Note 15

A *tournament* is a directed graph in which every pair of vertices has exactly one directed edge between them—for example, Fig. 1 has examples of two tournaments on the vertices $\{1,2,3\}$.

Figure 1: Examples of tournament graphs

In the first tournament above (Fig. 1a), $(1,2,3)$ is a *Hamiltonian path*, since it visits all the vertices exactly once, without repeating any edges, but $(1,2,3,1)$ is not a valid *Hamiltonian cycle*, because the tournament contains the directed edge $1 \to 3$ and not $3 \to 1$. In the second tournament (Fig. 1b), $(1,2,3,1)$ is a *Hamiltonian cycle*, as are $(2,3,1,2)$ and $(3,1,2,3)$; for this problem we'll say that these are all different Hamiltonian cycles, since their start/end points are different.

Consider the following way of choosing a random tournament T on n vertices: independently for each (unordered) pair of distinct vertices $\{i, j\} \subset \{1,\ldots,n\}$, flip a coin and include the edge $i \to j$ in the graph if the outcome is heads, and the edge $j \to i$ if tails.

(a) What is the expected number of Hamiltonian paths in T?

(b) What is the expected number of Hamiltonian cycles in T?

6 Swaps and Cycles

Note 15 We'll say that a permutation $\pi = (\pi(1), \ldots, \pi(n))$ contains a *swap* if there exist $i, j \in \{1, \ldots, n\}$ so that $\pi(i) = j$ and $\pi(j) = i$, where $i \neq j$.

(a) What is the expected number of swaps in a random permutation?

(b) In the same spirit as above, we'll say that π contains a *k-cycle* if there exist $i_1, \ldots, i_k \in \{1, \ldots, n\}$ with $\pi(i_1) = i_2, \pi(i_2) = i_3, \ldots, \pi(i_k) = i_1$. Compute the expectation of the number of k-cycles.