CS 70 Fall 2023 — Discrete Mathematics and Probability Theory — Tal, Rao — HW 12

Due: Saturday 4/22, 4:00 PM
Grace period until Saturday 4/22, 6:00 PM

Sundry

Before you start writing your final homework submission, state briefly how you worked on it. Who else did you work with? List names and email addresses. (In case of homework party, you can just describe the group.)

1 Tellers

Note 17

Imagine that X is the number of customers that enter a bank at a given hour. To simplify everything, in order to serve n customers you need at least n tellers. One less teller and you won't finish serving all of the customers by the end of the hour. You are the manager of the bank and you need to decide how many tellers there should be in your bank so that you finish serving all of the customers in time. You need to be sure that you finish in time with probability at least 95%.

(a) Assume that from historical data you have found out that $\mathbb{E}[X] = 5$. How many tellers should you have?

(b) Now assume that you have also found out that $\mathrm{Var}(X) = 5$. Now how many tellers do you need?

2 Just One Tail, Please

Note 17

Let X be some random variable with finite mean and variance which is not necessarily non-negative. The *extended* version of Markov's Inequality states that for a non-negative function $\varphi(x)$ which is monotonically increasing for $x > 0$ and some constant $\alpha > 0$,

$$\mathbb{P}[X \geq \alpha] \leq \frac{\mathbb{E}[\varphi(X)]}{\varphi(\alpha)}$$

Suppose $\mathbb{E}[X] = 0$, $\mathrm{Var}(X) = \sigma^2 < \infty$, and $\alpha > 0$.

(a) Use the extended version of Markov's Inequality stated above with $\varphi(x) = (x+c)^2$, where c is some positive constant, to show that:

$$\mathbb{P}[X \geq \alpha] \leq \frac{\sigma^2 + c^2}{(\alpha + c)^2}$$

(b) Note that the above bound applies for all positive c, so we can choose a value of c to minimize the expression, yielding the best possible bound. Find the value for c which will minimize the RHS expression (you may assume that the expression has a unique minimum).

We can plug in the minimizing value of c you found in part (b) to prove the following bound:

$$\mathbb{P}[X \geq \alpha] \leq \frac{\sigma^2}{\alpha^2 + \sigma^2}.$$

This bound is also known as Cantelli's inequality.

(c) Recall that Chebyshev's inequality provides a two-sided bound. That is, it provides a bound on $\mathbb{P}[|X - \mathbb{E}[X]| \geq \alpha] = \mathbb{P}[X \geq \mathbb{E}[X] + \alpha] + \mathbb{P}[X \leq \mathbb{E}[X] - \alpha]$. If we only wanted to bound the probability of one of the tails, e.g. if we wanted to bound $\mathbb{P}[X \geq \mathbb{E}[X] + \alpha]$, it is tempting to just divide the bound we get from Chebyshev's by two.

  (i) Provide an example of a random variable X (does not have to be zero-mean) and a constant α such that using this method (dividing by two to bound one tail) is not correct, that is, $\mathbb{P}[X \geq \mathbb{E}[X] + \alpha] > \frac{\mathrm{Var}(X)}{2\alpha^2}$ or $\mathbb{P}[X \leq \mathbb{E}[X] - \alpha] > \frac{\mathrm{Var}(X)}{2\alpha^2}$.

Now we see the use of the bound proven in part (b) - it allows us to bound just one tail while still taking variance into account, and does not require us to assume any property of the random variable. Note that the bound is also always guaranteed to be less than 1 (and therefore at least somewhat useful), unlike Markov's and Chebyshev's inequality!

(d) Let's try out our new bound on a simple example. Suppose X is a positively-valued random variable with $\mathbb{E}[X] = 3$ and $\mathrm{Var}(X) = 2$.

  (i) What bound would Markov's inequality give for $\mathbb{P}[X \geq 5]$?

  (ii) What bound would Chebyshev's inequality give for $\mathbb{P}[X \geq 5]$?

  (iii) What bound would Cantelli's Inequality give for $\mathbb{P}[X \geq 5]$? (*Note*: Recall that Cantelli's Inequality only applies for zero-mean random variables.)

3 Short Answer

Note 21

(a) Let X be uniform on the interval $[0,2]$, and define $Y = 4X^2 + 1$. Find the PDF, CDF, expectation, and variance of Y.

(b) Let X and Y have joint distribution

$$f(x,y) = \begin{cases} cxy + \frac{1}{4} & x \in [1,2] \text{ and } y \in [0,2] \\ 0 & \text{otherwise.} \end{cases}$$

Find the constant c (Hint: remember that the PDF must integrate to 1). Are X and Y independent?

(c) Let $X \sim \text{Exp}(3)$.

   (i) Find probability that $X \in [0, 1]$.

   (ii) Let $Y = \lfloor X \rfloor$, where the floor operator is defined as: $(\forall x \in [k, k+1))(\lfloor x \rfloor = k)$. For each $k \in \mathbb{N}$, what is the probability that $Y = k$? Write the distribution of Y in terms of one of the famous distributions; provide that distribution's name and parameters.

(d) Let $X_i \sim \text{Exp}(\lambda_i)$ for $i = 1, \ldots, n$ be mutually independent. It is a (very nice) fact that $\min(X_1, \ldots, X_n) \sim \text{Exp}(\mu)$. Find μ.

4 Uniform Distribution

Note 21 You have two fidget spinners, each having a circumference of 10. You mark one point on each spinner as a needle and place each of them at the center of a circle with values in the range $[0, 10)$ marked on the circumference. If you spin both (independently) and let X be the position of the first spinner's mark and Y be the position of the second spinner's mark, what is the probability that $X \geq 5$, given that $Y \geq X$?

5 Darts with Friends

Note 21 Michelle and Alex are playing darts. Being the better player, Michelle's aim follows a uniform distribution over a disk of radius 1 around the center. Alex's aim follows a uniform distribution over a disk of radius 2 around the center.

(a) Let the distance of Michelle's throw from the center be denoted by the random variable X and let the distance of Alex's throw from the center be denoted by the random variable Y.

   (i) What's the cumulative distribution function of X?

   (ii) What's the cumulative distribution function of Y?

   (iii) What's the probability density function of X?

   (iv) What's the probability density function of Y?

(b) What's the probability that Michelle's throw is closer to the center than Alex's throw? What's the probability that Alex's throw is closer to the center?

(c) What's the cumulative distribution function of $U = \max(X, Y)$?