CS 70 Fall 2023 — Discrete Mathematics and Probability Theory — Tal, Rao — HW 14

Due: Saturday 12/2, 4:00 PM
Grace period until Saturday 12/2, 6:00 PM

Sundry

Before you start writing your final homework submission, state briefly how you worked on it. Who else did you work with? List names and email addresses. (In case of homework party, you can just describe the group.)

1 Moments of the Gaussian

Note 21

For a random variable X, the quantity $\mathbb{E}[X^k]$ for $k \in \mathbb{N}$ is called the *kth moment* of the distribution. In this problem, we will calculate the moments of a standard normal distribution.

(a) Prove the identity

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\left(-\frac{tx^2}{2}\right) \mathrm{d}x = t^{-1/2}$$

for $t > 0$.

Hint: Consider a normal distribution with variance $\frac{1}{t}$ and mean 0.

(b) For the rest of the problem, X is a standard normal distribution (with mean 0 and variance 1). Use part (a) to compute $\mathbb{E}[X^{2k}]$ for $k \in \mathbb{N}$.

Hint: Try differentiating both sides with respect to t, k times. You may use the fact that we can differentiate under the integral without proof.

(c) Compute $\mathbb{E}[X^{2k+1}]$ for $k \in \mathbb{N}$.

2 Chebyshev's Inequality vs. Central Limit Theorem

Note 17
Note 21

Let n be a positive integer. Let $X_1, X_2, \ldots, X_n$ be i.i.d. random variables with the following distribution:

$$\mathbb{P}[X_i = -1] = \frac{1}{12}; \qquad \mathbb{P}[X_i = 1] = \frac{9}{12}; \qquad \mathbb{P}[X_i = 2] = \frac{2}{12}.$$

(a) Calculate the expectations and variances of X_1, $\sum_{i=1}^{n} X_i$, $\sum_{i=1}^{n}(X_i - \mathbb{E}[X_i])$, and

$$Z_n = \frac{\sum_{i=1}^{n}(X_i - \mathbb{E}[X_i])}{\sqrt{n/2}}.$$

(b) Use Chebyshev's Inequality to find an upper bound b for $\mathbb{P}[|Z_n| \geq 2]$.

(c) Use b from the previous part to bound $\mathbb{P}[Z_n \geq 2]$ and $\mathbb{P}[Z_n \leq -2]$.

(d) As $n \to \infty$, what is the distribution of Z_n?

(e) We know that if $Z \sim \mathcal{N}(0,1)$, then $\mathbb{P}[|Z| \leq 2] = \Phi(2) - \Phi(-2) \approx 0.9545$. As $n \to \infty$, provide approximations for $\mathbb{P}[Z_n \geq 2]$ and $\mathbb{P}[Z_n \leq -2]$.

3 Continuous LLSE

Note 20

Suppose that X and Y are uniformly distributed on the shaded region in the figure below.

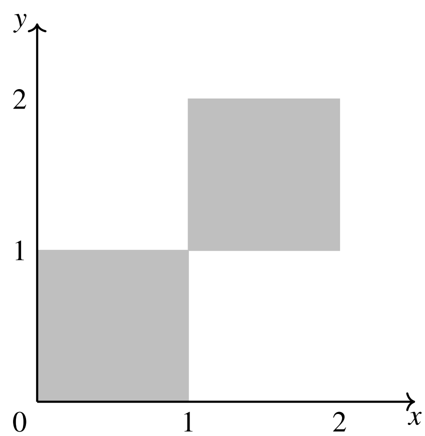


Figure 1: The joint density of (X,Y) is uniform over the shaded region.

That is, X and Y have the joint distribution:

$$f_{X,Y}(x,y) = \begin{cases} 1/2, & 0 \leq x \leq 1,\ 0 \leq y \leq 1 \\ 1/2, & 1 \leq x \leq 2,\ 1 \leq y \leq 2 \end{cases}$$

(a) Do you expect X and Y to be positively correlated, negatively correlated, or neither?

(b) Compute the marginal distribution of X.

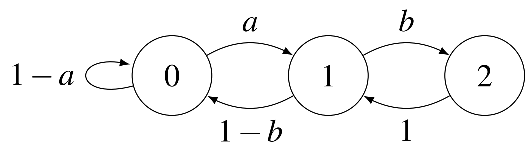
(c) Compute $L[Y \mid X]$, the best linear estimator of Y given X.

(d) What is $\mathbb{E}[Y \mid X]$?

4 Analyze a Markov Chain

Note 22

Consider a Markov chain with the state diagram shown below where $a, b \in (0,1)$.



Here, we let $X(n)$ denote the state at time n.

(a) Show that this Markov chain is aperiodic.

(b) Calculate $\mathbb{P}[X(1) = 1, X(2) = 0, X(3) = 0, X(4) = 1 \mid X(0) = 0]$.

(c) Calculate the invariant distribution.

(d) What is the expected number of steps we need to take before we reach state 2, given that we start in state 1?

5 Playing Blackjack

Note 22

You are playing a game of Blackjack where you start with \$100. You are a particularly risk-loving player who does not believe in leaving the table until you either make \$400, or lose all your money. At each turn you either win \$100 with probability $\frac{1}{2}$, or you lose \$100 with probability $\frac{1}{2}$.

(a) Formulate this problem as a Markov chain; i.e. define your state space, transition probabilities, and determine your starting state.

(b) Let $\alpha(i)$ denote the probability that you end the game with \$400, given that you started with i dollars. Write a system of equations involving $\alpha(i)$ for $i \in \{0, 100, 200, 300, 400\}$.

(c) Given that you started with \$100, what is the probability that you end the game with \$400? What about if you started with \$200? What about if you started with \$300?

Let us now generalize the above scenario. Suppose you start with \$1, and at each turn, you win \$1 with probability p, or lose \$1 with probability $1 - p$. You will continually play games of Blackjack until you either lose all your money, or you have a total of n dollars.

(d) Formulate this problem as a Markov chain.

(e) Let $\alpha(i)$ denote the probability that you end the game with n dollars, given that you started with i dollars.

Notice that for $0 < i < n$, we can write $\alpha(i+1) - \alpha(i) = k(\alpha(i) - \alpha(i-1))$. Find k.

(f) Using part (e), find $\alpha(i)$, where $0 \leq i \leq n$. (You will need to split into two cases: $p = \frac{1}{2}$ or $p \neq \frac{1}{2}$.)

Hint: Try to apply part (e) iteratively, and look at a telescoping sum to write $\alpha(i)$ in terms of $\alpha(1)$. The formula for the sum of a finite geometric series may be helpful when looking at the case where $p \neq \frac{1}{2}$:

$$\sum_{k=0}^{m} a^k = \frac{1 - a^{m+1}}{1 - a}.$$

Lastly, it may help to use the value of $\alpha(n)$ to find $\alpha(1)$ for the last few steps of the calculation.

(g) As $n \to \infty$, what happens to the probability of ending the game with n dollars, given that you start with i dollars, with the following values of p?

(i) $p > \frac{1}{2}$

(ii) $p = \frac{1}{2}$

(iii) $p < \frac{1}{2}$