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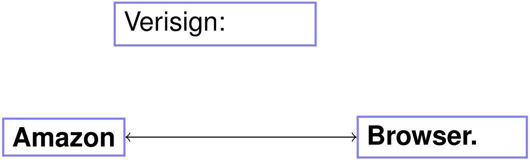

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Amazon $\longleftrightarrow$ **Browser.**

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Polynomials.

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Secret Sharing.

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Correcting for loss or even corruption.

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Lots of lines go through one point.

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A **polynomial**

$$P(x) = a_d x^d + a_{d-1} x^{d-1} \cdots + a_0.$$

is specified by **coefficients** $a_d, \ldots a_0$.

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[2] A field is a set of elements with addition and multiplication operations, with inverses. $GF(p) = (\{0, \ldots, p-1\}, + \pmod p, * \pmod p)$.

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Polynomials $P(x)$ **with arithmetic modulo** p: [2] $a_i \in \{0, \ldots, p-1\}$ and

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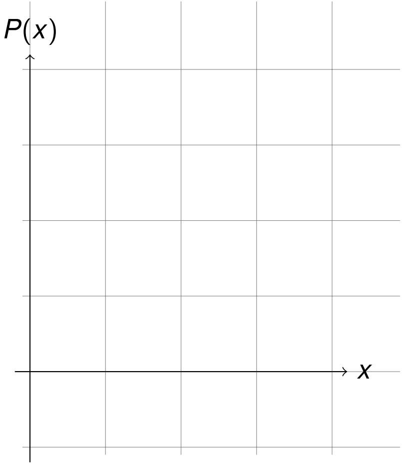

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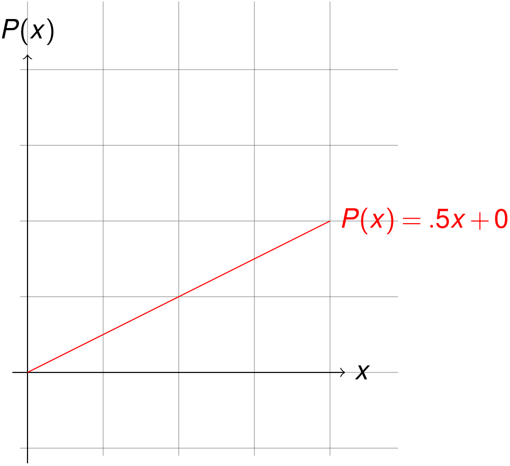

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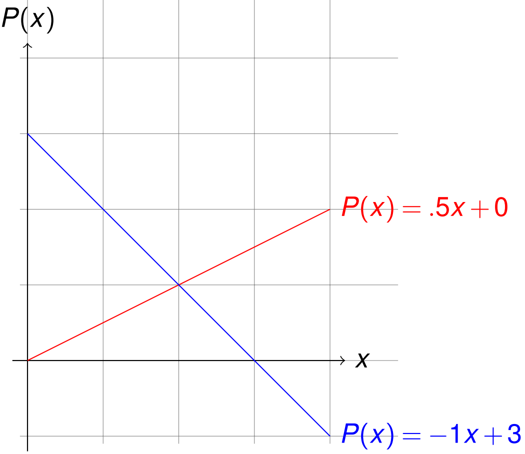

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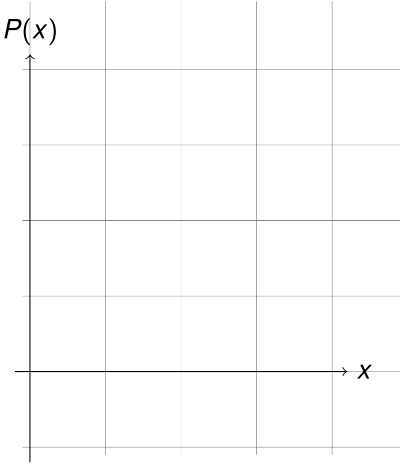

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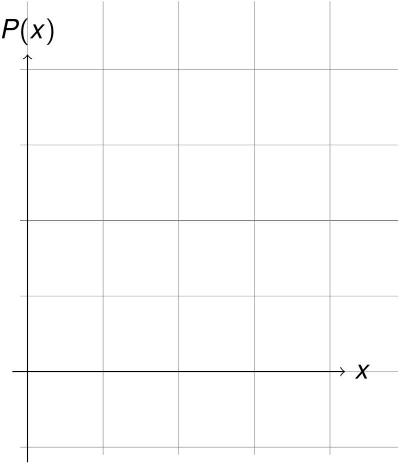

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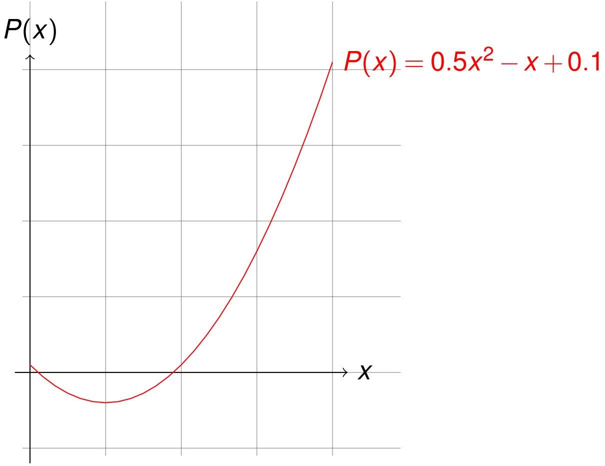

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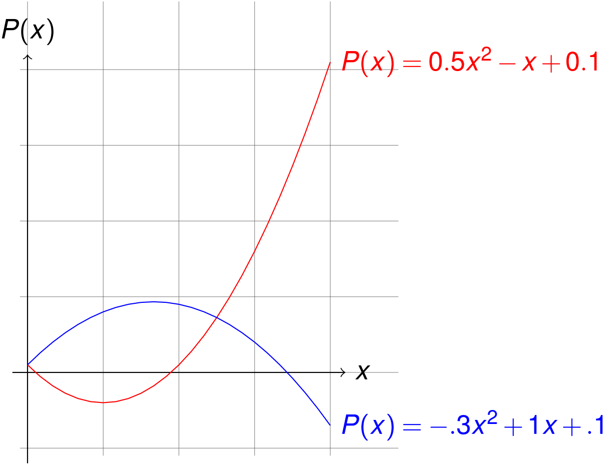

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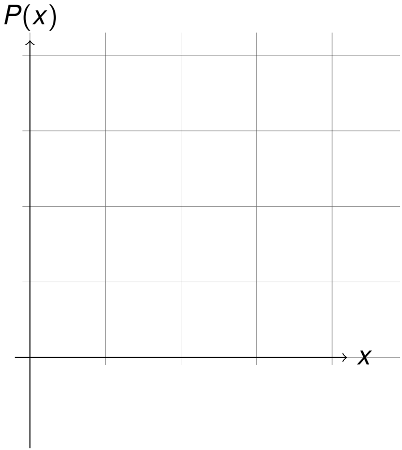

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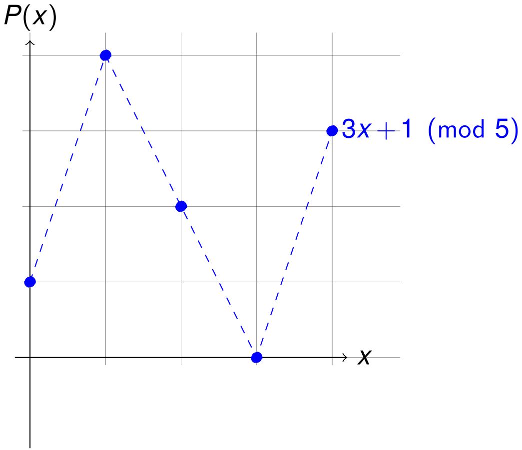

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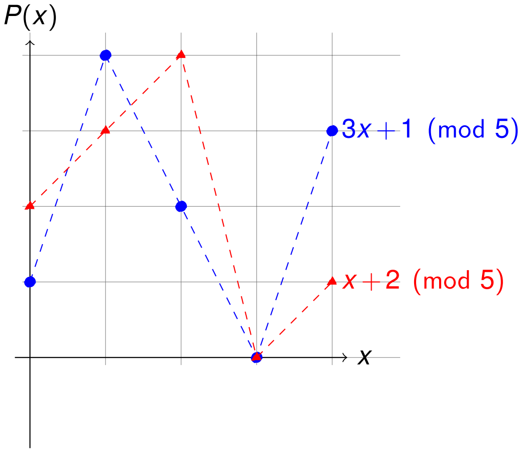

Finding an intersection.
$x + 2 \equiv 3x + 1 \pmod 5$
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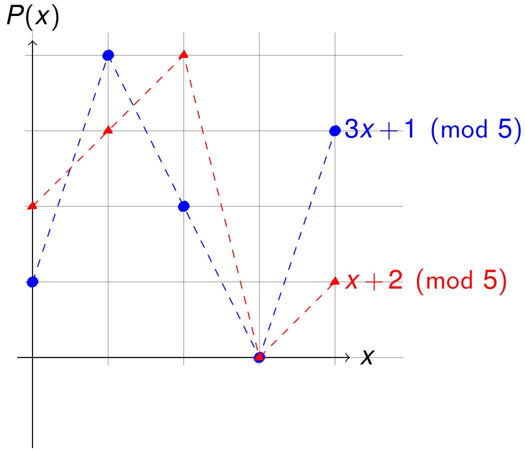

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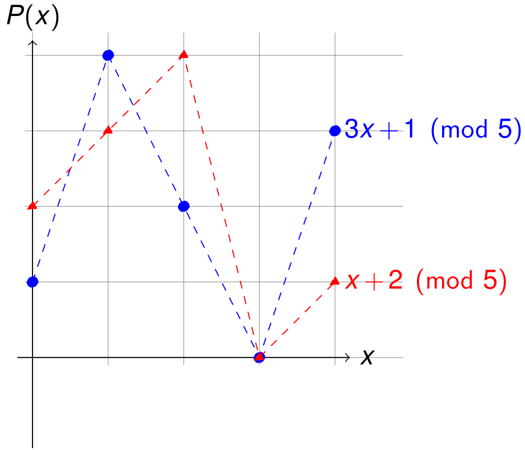

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Good when modulus is prime!!

Two points make a line.

Fact: Exactly 1 degree $\leq d$ polynomial contains $d + 1$ points. [3]

[3] Points with different x values.

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Poll.

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What facts below tell you this?

Say points are $(x_1, y_1), (x_2, y_2)$**.**

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(A) Line is $y = mx + b$.
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All true.

In the Flow (Steph Curry) Poll.

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(A) Solution cuz: $m = (y_2 - y_1)/(x_2 - x_1), b = y_1 - m(x_1)$
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(C) Try: $(m'x + b') - (mx + b) = (m' - m)x + (b - b') = ax + c \neq 0$.
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Flow poll. (All true. (B) is not a proof, it is restatement.)

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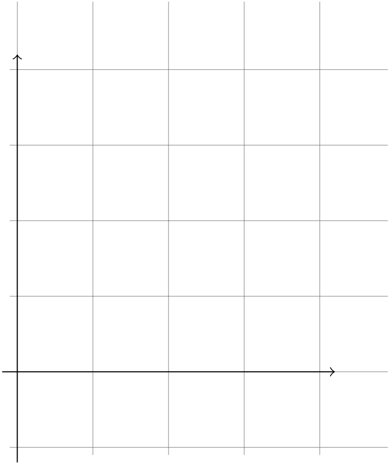

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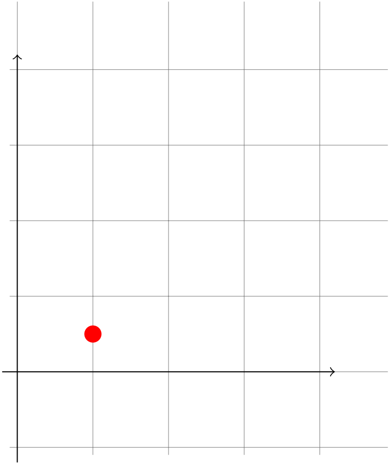

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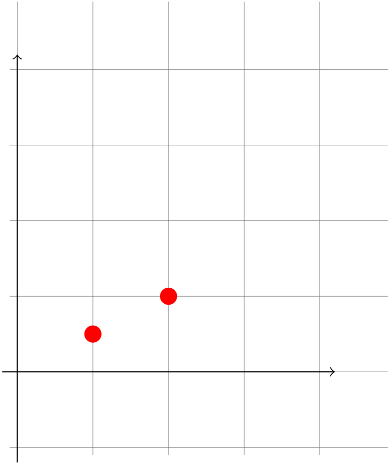

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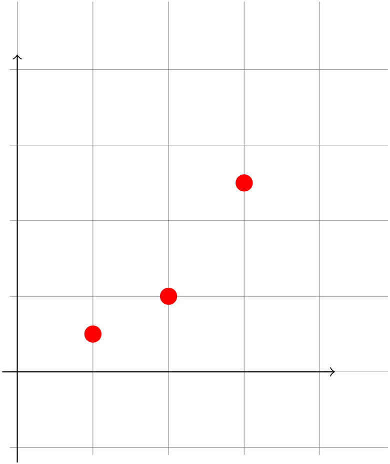

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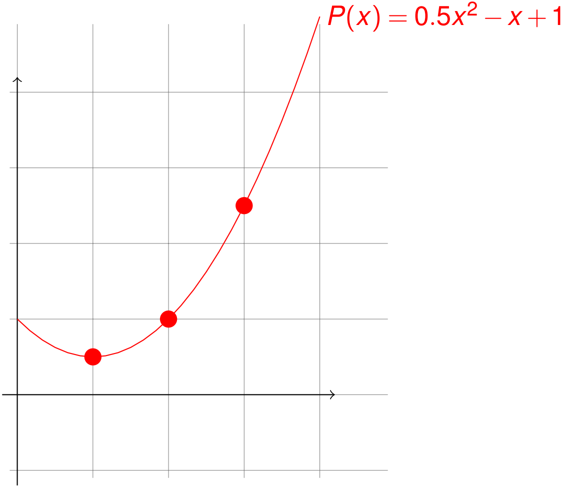

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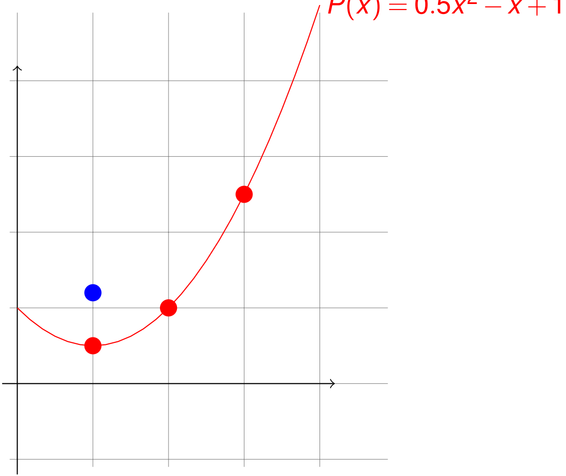

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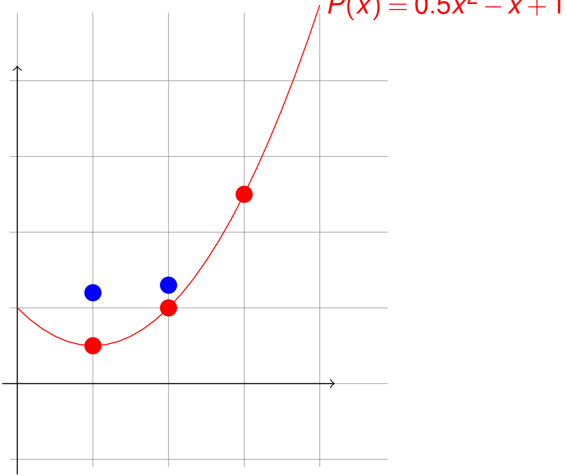

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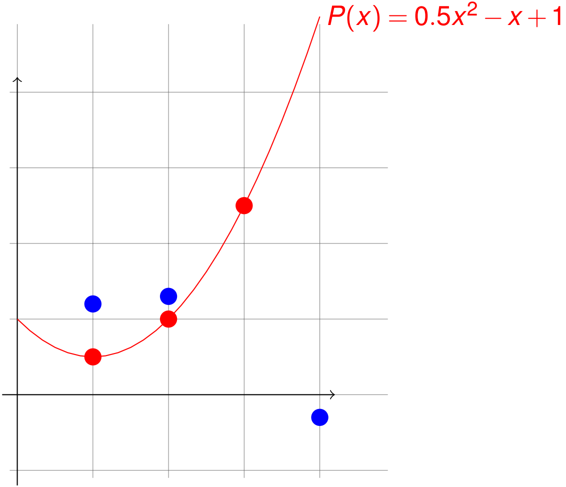

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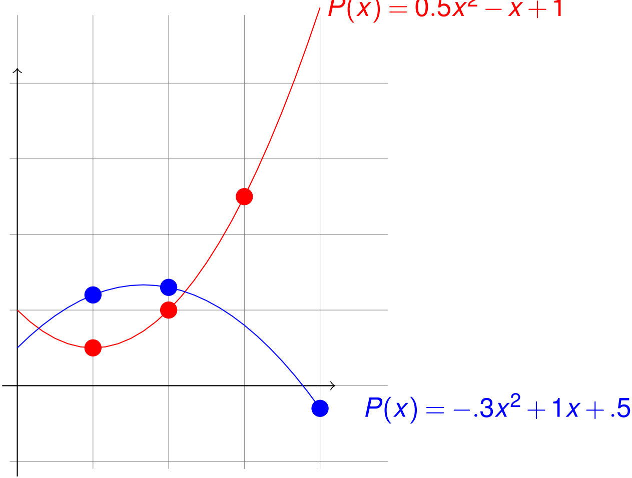

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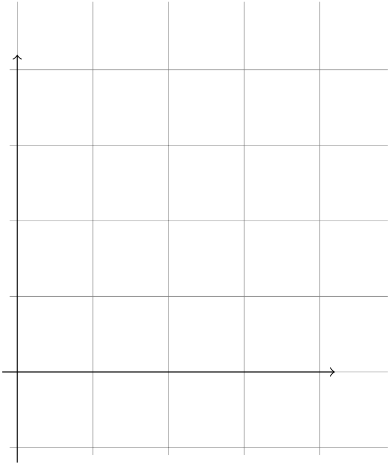

There is $P(x)$ contains blue points and *any* $(0, y)$!

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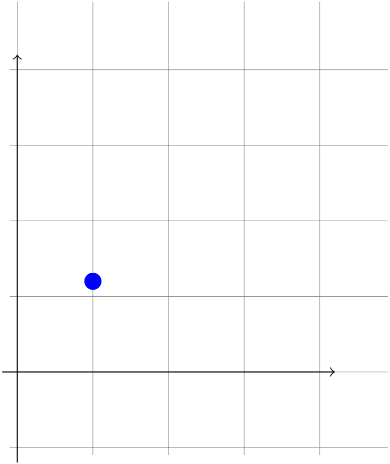

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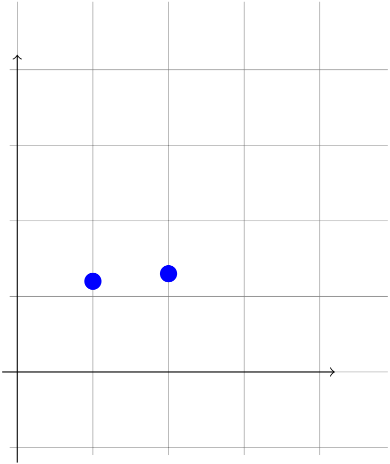

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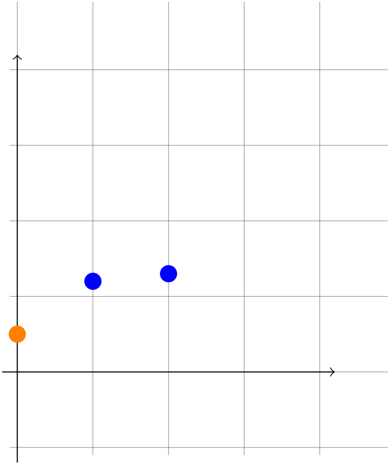

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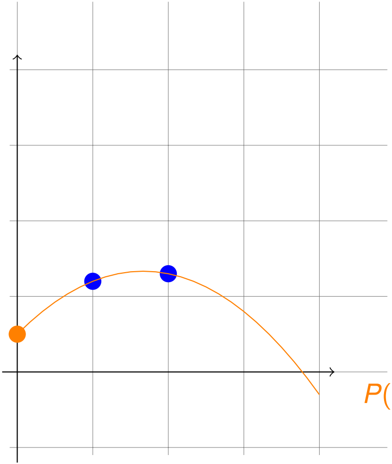

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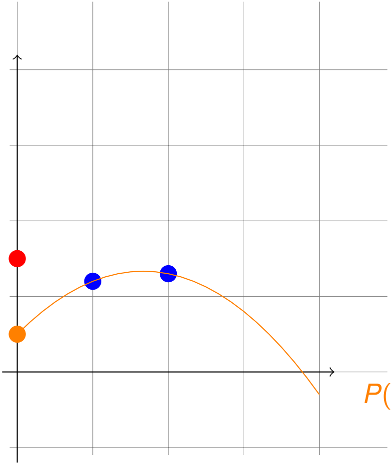

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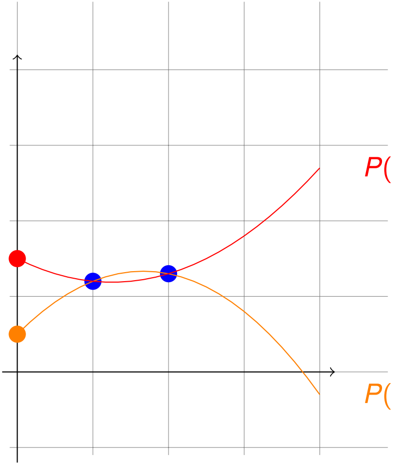

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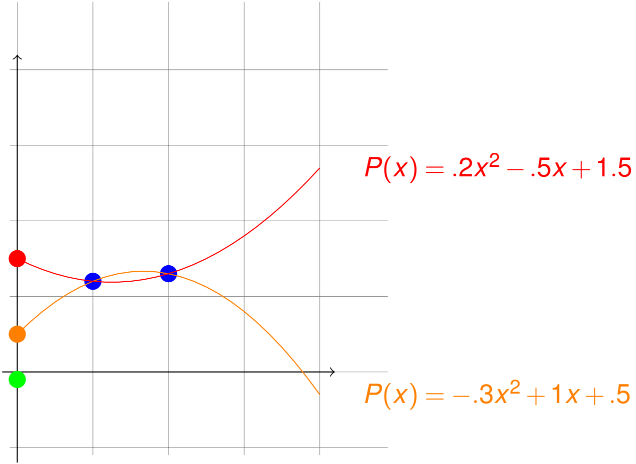

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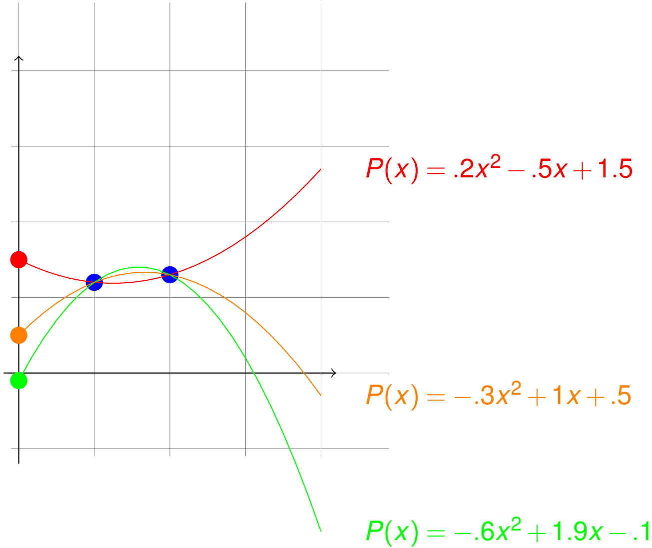

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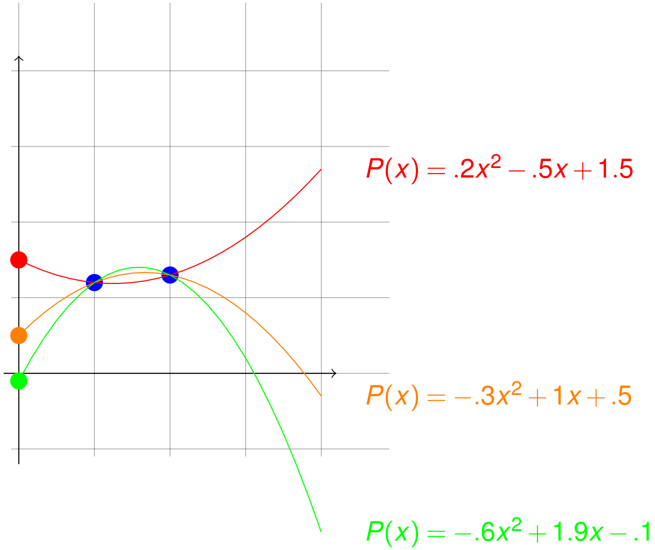

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Poll:example.

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But wanted to hit $(1,2);(2,4);(3,0)$!

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Another Construction: Interpolation!

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Multiplicative inverses due to $gcd(x,p) = 1$, forall $x \in \{1,\ldots,p-1\}$

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Construction proves the existence of the polynomial!

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Uniqueness Fact. At most one degree d polynomial hits $d+1$ points.

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Must prove **Roots fact.**

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That is, $P(x) = (x - a)Q(x) + r$

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Intuitively, a field is a set with operations corresponding to addition, multiplication, and division.

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Modular Arithmetic Fact: Exactly one polynomial degree $\leq d$ over $GF(p)$, $P(x)$, that hits $d+1$ points.

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Secret $s \in \{0,\ldots,p-1\}$

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(Almost) the same as what is missing: one $P(i)$.

Runtime.

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Runtime: polynomial in k, n, and $\log p$.

1. Evaluate degree $k - 1$ polynomial n times using $\log p$-bit numbers.
2. Reconstruct secret by solving system of k equations using $\log p$-bit arithmetic.

A bit more counting.

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Infinite number for reals, rationals, complex numbers!

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Compute solution: m, b.

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- $P(x) - Q(x)$ is degree $d \implies d$ roots.
- $P(x) = Q(x)$ on $d+1$ points $\implies P(x) = Q(x)$.

Secret Sharing:

- k points on degree $k-1$ polynomial is great!
- Can hand out n points on polynomial as shares.