Today.

Finish up counting.

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Countabiity.

Some Practice.

How many orderings of letters of CAT?

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Ordered, except for A!

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total orderings of 7 letters.

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11 letters total.

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Summary.

First rule: $n_1 \times n_2 \cdots \times n_3$.

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Sample k times from n objects with replacement and order doesn't matter: $\binom{k+n-1}{n-1}$.

Sampling...

Sample k items out of n

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Without replacement:

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Without replacement:

Order matters:

Sampling...

Sample k items out of n

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Order matters: $n \times$

Sampling...

Sample k items out of n

Without replacement:

Order matters: $n \times n-1 \times n-2 \dots$

Sampling...

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Order matters: $n \times n-1 \times n-2 \ldots \times n-k+1$

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Order matters: $n \times n-1 \times n-2 \ldots \times n-k+1 = \frac{n!}{(n-k)!}$

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Second Rule: divide by number of orders

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With Replacement.

Order matters: $n \times n$

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Order matters: $n \times n \times \ldots n = n^k$

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Different number of unordered elts map to each unordered elt.

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How do we deal with this mess??

Splitting up some money....

How many ways can Bob and Alice split 5 dollars?

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“Sorted” way to specify, first Alice’s dollars, then Bob’s.

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and so on.

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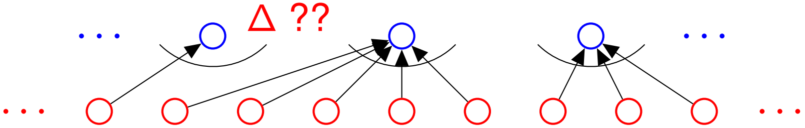

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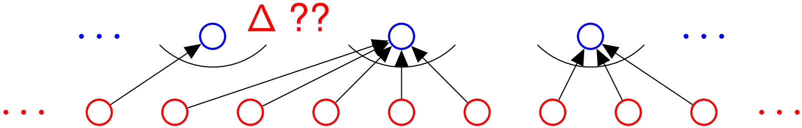

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"Sorted" way to specify, first Alice's dollars, then Bob's.

(B, B, B, B, B): 1: (B,B,B,B,B)
(A, B, B, B, B): 5: (A,B,B,B,B),(B,A,B,B,B),(B,B,A,B,B),...
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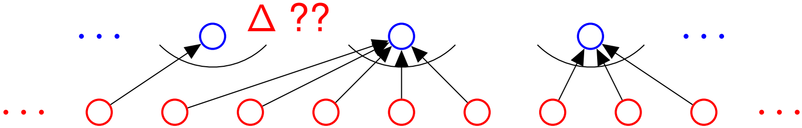

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For each of 5 dollars pick Bob or Alice(2^5), divide out order ???

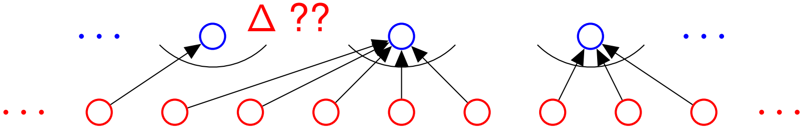
5 dollars for Bob and 0 for Alice:
one ordered set: (B, B, B, B, B).

4 for Bob and 1 for Alice:
5 ordered sets: (A, B, B, B, B) ; (B, A, B, B, B); ...

"Sorted" way to specify, first Alice's dollars, then Bob's.

(B, B, B, B, B): 1: (B,B,B,B,B)
(A, B, B, B, B): 5: (A,B,B,B,B),(B,A,B,B,B),(B,B,A,B,B),...
(A, A, B, B, B): $\binom{5}{2}$;(A,A,B,B,B),(A,B,A,B,B),(A,B,B,A,B),...

and so on.



Second rule of counting is no good here!

Splitting 5 dollars..

How many ways can Alice, Bob, and Eve split 5 dollars.

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Separate Alice's dollars from Bob's and then Bob's from Eve's.

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Counting Rule: if there is a one-to-one mapping between two sets they have the same size!

Stars and Bars.

How many different 5 star and 2 bar diagrams?

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Bars in second and seventh position.

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$\binom{7}{2}$ ways to do so and
$\binom{7}{2}$ ways to split 5 dollars among 3 people.

Stars and Bars.

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Or: k unordered choices from set of n possibilities with replacement.
Sample with replacement where order doesn't matter.

Counting basics.

First rule: $n_1 \times n_2 \cdots \times n_3$.

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Sample without replacement and order doesn't matter: $\binom{n}{k} = \frac{n!}{(n-k)!k!}$.
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One-to-one rule: equal in number if one-to-one correspondence.

Sample with replacement and order doesn't matter: $\binom{k+n-1}{n-1}$.

Bijection: sums to 'k' $\rightarrow$ stars and bars.

$$S = \{(n_1, n_2, n_3) : n_1 + n_2 + n_3 = 5\}$$

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$|S| = |T| = \binom{7}{2}$.

Stars and Bars Poll

Mark whats correct.

(A) ways to split n dollars among k: $\binom{n+k-1}{k-1}$

(B) ways to split k dollars among n: $\binom{k+n-1}{n-1}$

(C) ways to split 5 dollars among 3: $\binom{7}{5}$

(D) ways to split 5 dollars among 3: $\binom{5+3-1}{3-1}$

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All correct.

Balls in bins.

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Example: 5 digit numbers.

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Example: Poker hands.

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5 balls into 10 bins
5 samples from 10 possibilities with replacement
Example: 5 digit numbers.

5 indistinguishable balls into 52 bins only one ball in each bin
5 samples from 52 possibilities without replacement
Example: Poker hands.

5 indistinguishable balls into 3 bins
5 samples from 3 possibilities with replacement and no order

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5 balls into 10 bins
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Example: 5 digit numbers.

5 indistinguishable balls into 52 bins only one ball in each bin
5 samples from 52 possibilities without replacement
Example: Poker hands.

5 indistinguishable balls into 3 bins
5 samples from 3 possibilities with replacement and no order
Dividing 5 dollars among Alice, Bob and Eve.

Poll

Mark whats correct.

k Balls in n bins.

dis == distinguishiable
unique = one ball in each bin.

(A) dis $=>$ n^k
(B) dis,unique $=>$ $n!/(n-k)!$
(C) indis, unique $=>$ $\binom{n}{k}$
(D) dis, $=>$ $n!/(n-k)!$
(E) indis, $=>$ $\binom{n+k-1}{k-1}$
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Sum Rule

Two indistinguishable jokers in 54 card deck.
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How many 5 card poker hands? Choose 4 cards plus one of 2 jokers!

$$\binom{52}{5} + 2 * \binom{52}{4} +$$

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Theorem: $\binom{54}{5}$

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Theorem: $\binom{54}{5} = \binom{52}{5} + 2 * \binom{52}{4} + \binom{52}{3}$.

Algebraic Proof:

Sum Rule

Two indistinguishable jokers in 54 card deck.
How many 5 card poker hands?
Sum rule: Can sum over disjoint sets.
No jokers "exclusive" or One Joker "exclusive" or Two Jokers

$$\binom{52}{5} + \binom{52}{4} + \binom{52}{3}.$$

Two distinguishable jokers in 54 card deck.
How many 5 card poker hands? Choose 4 cards plus one of 2 jokers!

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Theorem: $\binom{n}{k} = \binom{n}{n-k}$

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□

Pascal’s Triangle

Pascal's Triangle

0

1 1

Pascal’s Triangle

0
1 1
1 2 1

Pascal's Triangle

			0			
		1		1		
	1		2		1	
1		3		3		1

Pascal's Triangle

				0				
			1		1			
		1		2		1		
	1		3		3		1	
1		4		6		4		1

Pascal's Triangle

0
1 1
1 2 1
1 3 3 1
1 4 6 4 1

Pascal's Triangle

				0				
			1		1			
		1		2		1		
	1		3		3		1	
1		4		6		4		1

Row n: coefficients of $(1+x)^n = (1+x)(1+x)\cdots(1+x)$.

Pascal's Triangle

$$\begin{array}{ccccccccc} & & & & 0 & & & & \\ & & & 1 & & 1 & & & \\ & & 1 & & 2 & & 1 & & \\ & 1 & & 3 & & 3 & & 1 & \\ 1 & & 4 & & 6 & & 4 & & 1 \end{array}$$

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Foil (4 terms)

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Foil (4 terms) on steroids:

Pascal's Triangle

				0				
			1		1			
		1		2		1		
	1		3		3		1	
1		4		6		4		1

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 & & & & 0 & & & & \\
 & & & 1 & & 1 & & & \\
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0

1 1

1 2 1

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Simplify: collect all terms corresponding to x^k.

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  Coefficient of x^k $\binom{n}{k}$: choose k terms with x in product.

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$$
\begin{array}{ccccccccc}
 & & & & 0 & & & & \\
 & & & 1 & & 1 & & & \\
 & & 1 & & 2 & & 1 & & \\
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$$
\begin{array}{ccc}
 & \binom{0}{0} & \\
\binom{1}{0} & & \binom{1}{1}
\end{array}
$$

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\end{array}
$$

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 & & & & 0 & & & & \\
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Pascal's rule $\implies \binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}$.

Combinatorial Proofs.

Theorem: $\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}$.

Proof: How many size k subsets of $n+1$?

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How many size k subsets of $n+1$?
How many contain the first element?

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How many size k subsets of $n+1$?
How many contain the first element?
Chose first element, need $k-1$ more from remaining n elements.

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Sum Rule: size of union of disjoint sets of objects.

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Without and with first element $\rightarrow$ disjoint.

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Sum Rule: size of union of disjoint sets of objects.
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So, $\binom{n}{k-1} + \binom{n}{k} = \binom{n+1}{k}$. $\square$

Combinatorial Proof.

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$$\{1, \ldots, i, \underline{\ldots, n}\}$$

Must choose $k-1$ elements from $n-i$ remaining elements.

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Add them up to get the total number of subsets of size k
which is also $\binom{n+1}{k}$.

$\square$

Binomial Theorem: $x = 1$

Theorem: $2^n = \binom{n}{n} + \binom{n}{n-1} + \cdots + \binom{n}{0}$

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Construct a subset with sequence of n choices:

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element i **is in** or **is not** in the subset: 2 poss.

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Sum over i to get total number of subsets..

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Sum over i to get total number of subsets..which is also 2^n. □

Simple Inclusion/Exclusion

Sum Rule: For disjoint sets S and T, $|S \cup T| = |S| + |T|$
Used to reason about all subsets
by adding number of subsets of size 1, 2, 3,...

Simple Inclusion/Exclusion

Sum Rule: For disjoint sets S and T, $|S \cup T| = |S| + |T|$
Used to reason about all subsets
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Also reasoned about subsets that contained
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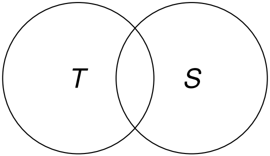

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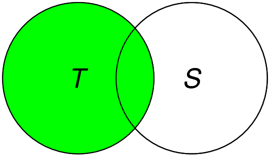

In T. $\implies |T|$

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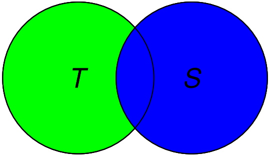

In T. $\implies$ $|T|$
In S. $\implies$ $+$ $|S|$

Simple Inclusion/Exclusion

Sum Rule: For disjoint sets S and T, $|S \cup T| = |S| + |T|$
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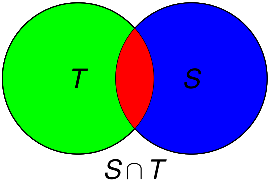

In T. $\implies$ $|T|$
In S. $\implies$ $+$ $|S|$
Elements in $S \cap T$ are counted twice.

Simple Inclusion/Exclusion

Sum Rule: For disjoint sets S and T, $|S \cup T| = |S| + |T|$
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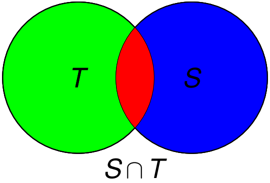

In T. $\implies |T|$
In S. $\implies + \; |S|$
Elements in $S \cap T$ are counted twice.
Subtract. $\implies -|S \cap T|$

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Sum Rule: For disjoint sets S and T, $|S \cup T| = |S| + |T|$
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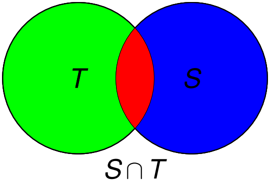

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Example: How many 10-digit phone numbers have 7 as their first or second digit?

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Example: How many 10-digit phone numbers have 7 as their first or second digit?

S = phone numbers with 7 as first digit.

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Example: How many 10-digit phone numbers have 7 as their first or second digit?

S = phone numbers with 7 as first digit. $|S| = 10^9$

Simple Inclusion/Exclusion

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Example: How many 10-digit phone numbers have 7 as their first or second digit?

S = phone numbers with 7 as first digit. $|S| = 10^9$

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T = phone numbers with 7 as second digit. $|T| = 10^9$.

$S \cap T$ = phone numbers with 7 as first and second digit. $|S \cap T| = 10^8$.

Answer: $|S| + |T| - |S \cap T| = 10^9 + 10^9 - 10^8$.

Inclusion/Exclusion

$|A_1 \cup \cdots \cup A_n| =$
$\sum_i |A_i| - \sum_{i,j} |A_i \cap A_j| + \sum_{i,j,k} |A_i \cap A_j \cap A_k| \cdots (-1)^n |A_1 \cap \cdots A_n|$.

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Idea: For $n = 3$ how many times is an element counted?

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Total: 3 - 3 + 1 = 1.

Inclusion/Exclusion

$|A_1 \cup \cdots \cup A_n| =$
$\sum_i |A_i| - \sum_{i,j} |A_i \cap A_j| + \sum_{i,j,k} |A_i \cap A_j \cap A_k| \cdots (-1)^n |A_1 \cap \cdots A_n|$.

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Total: 2 -1 = 1.

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Total: 3 - 3 + 1 = 1.

Formulaically: x is in intersection of three sets.

Inclusion/Exclusion

$|A_1 \cup \cdots \cup A_n| =$
$\sum_i |A_i| - \sum_{i,j} |A_i \cap A_j| + \sum_{i,j,k} |A_i \cap A_j \cap A_k| \cdots (-1)^n |A_1 \cap \cdots A_n|$.

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Total: 2 -1 = 1.

Consider $x \in A_1 \cap A_2 \cap A_3$

x counted once in each term: $|A_1|, |A_2|, |A_3|$.

x subtracted once in terms: $|A_1 \cap A_3|$, $|A_1 \cap A_2|$, $|A_2 \cap A_3|$.

x added once in $|A_1 \cap A_2 \cap A_3|$.

Total: 3 - 3 + 1 = 1.

Formulaically: x is in intersection of three sets.

3 for terms of form $|A_i|$, $\binom{3}{2}$ for terms of form $|A_i \cap A_j|$.

Inclusion/Exclusion

$|A_1 \cup \cdots \cup A_n| =$
$\sum_i |A_i| - \sum_{i,j} |A_i \cap A_j| + \sum_{i,j,k} |A_i \cap A_j \cap A_k| \cdots (-1)^n |A_1 \cap \cdots A_n|$.

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Consider $x \in A_i \cap A_j$.

x counted once for $|A_i|$ and once for $|A_j|$.

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Total: 2 -1 = 1.

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x counted once in each term: $|A_1|, |A_2|, |A_3|$.

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Inclusion/Exclusion

$|A_1 \cup \cdots \cup A_n| =$
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- x added once in $|A_1 \cap A_2 \cap A_3|$.

Total: 3 - 3 + 1 = 1.

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3 for terms of form $|A_i|$, $\binom{3}{2}$ for terms of form $|A_i \cap A_j|$.
$\binom{3}{3}$ for terms of form $|A_i \cap A_j \cap A_k|$.
Total: $\binom{3}{1} - \binom{3}{2} + \binom{3}{3} = 1$.

Inclusion/Exclusion

$|A_1 \cup \cdots \cup A_n| =$
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Idea: how many times is each element counted?

Element x in m sets: $x \in A_{i_1} \cap A_{i_2} \cdots \cap A_{i_m}$.

Inclusion/Exclusion

$|A_1 \cup \cdots \cup A_n| =$
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Counted $\binom{m}{i}$ times in ith summation.

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- Total: $\binom{m}{1} - \binom{m}{2} + \binom{m}{3} \cdots + (-1)^{m-1}\binom{m}{m}$.

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Each element counted once!

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Disjoint – so add!

Poll: How big is infinity?

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Mark what's true.
(A) There are more real numbers than natural numbers.
(B) There are more rational numbers than natural numbers.
(C) There are more integers than natural numbers.
(D) pairs of natural numbers $>>$ natural numbers.

Same Size. Poll.

Two sets are the same size?

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(A) Bijection between the sets.
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(A), (B).
(C)?

Countable.

How to count?

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How to count?

0,

Countable.

How to count?

0, 1,

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How to count?

0, 1, 2,

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How to count?

0, 1, 2, 3,

Countable.

How to count?

0, 1, 2, 3, ...

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Definition: S is **countable** if there is a bijection between S and some subset of N.

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If the subset of N is finite, S has finite **cardinality**.

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Definition: S is **countable** if there is a bijection between S and some subset of N.

If the subset of N is finite, S has finite **cardinality**.

If the subset of N is infinite, S is **countably infinite**.

Countably infinite subsets.

Enumerating a set implies countable.
Corollary: Any subset T of a countable set S is countable.

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All countably infinite sets have the same cardinality.

Enumeration example.

All binary strings.

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$B = \{0, 1\}^*$.

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Enumeration example.

All binary strings.
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Never get to 1.

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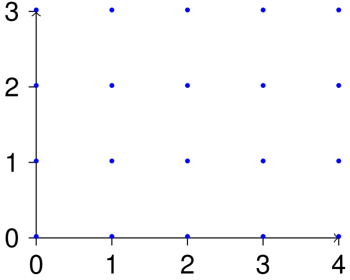

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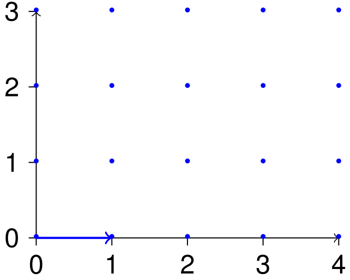

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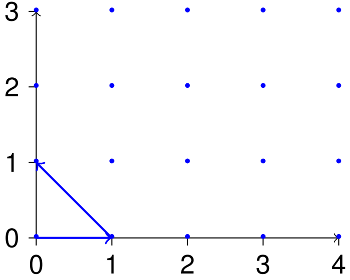

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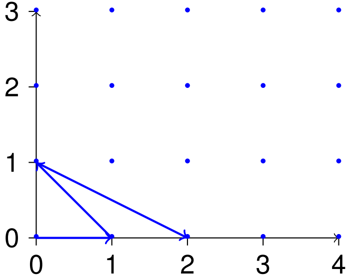

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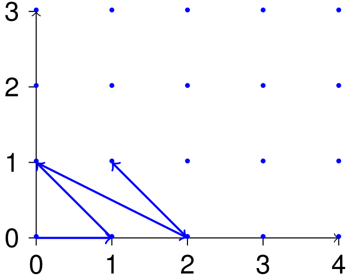

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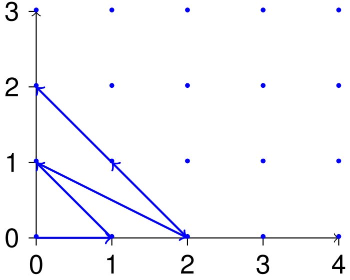

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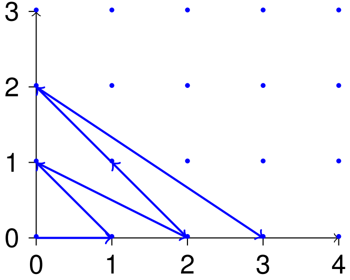

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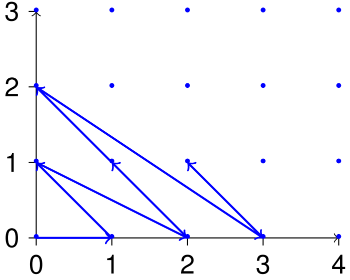

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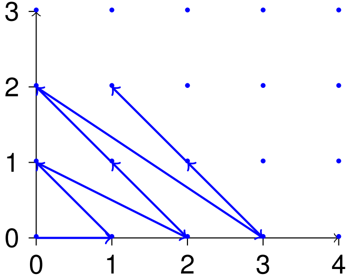

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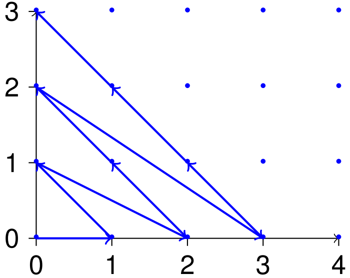

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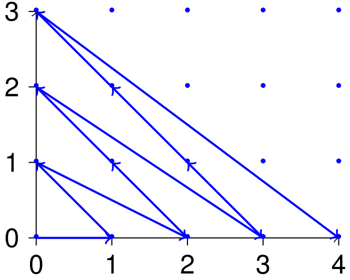

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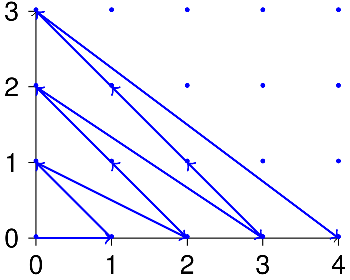

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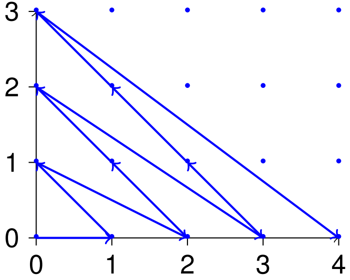

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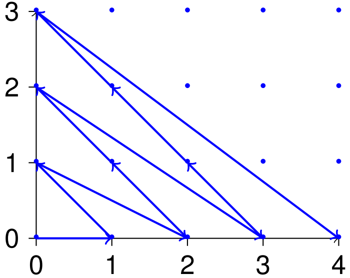

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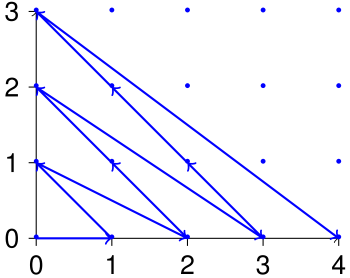

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Same size as the natural numbers!!

Poll.

Enumeration to get bijection with naturals?

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(A) Integers: First all negatives, then positives.
(B) Integers: By absolute value, break ties however.
(C) Pairs of naturals: by sum of values, break ties however.
(D) Pairs of naturals: by value of first element.
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(F) Pairs of integers: by sum of absolute values, break ties.

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(B),(C), (F).

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Real numbers..

Real numbers are same size as integers?

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Diagonalization.

If countable, there a listing, *L* contains all reals. For example

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1: .785398162...
2: .367879441...

Diagonalization.

If countable, there a listing, L contains all reals. For example

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Construct "diagonal" number:

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0: .500000000...
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Construct "diagonal" number: .77

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Construct "diagonal" number: .776

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Diagonal Number: Digit i is 7 if number i's ith digit is not 7

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If countable, there a listing, L contains all reals. For example

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Subset $[0,1]$ is not countable!!

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If reals are countable then so is $[0,1]$.

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Another diagonalization.

The set of all subsets of N.

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Example subsets of N: $\{0\}$,

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(The set of all subsets of S, is the **powerset** of N.)

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(A) Integers are larger than naturals cuz obviously.
(B) Integers are countable cuz, interleaving bijection.
(C) Reals are uncountable cuz obviously!
(D) Reals can't be in a list: diagonal number not on list.
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The Continuum hypothesis.

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First of Hilbert's problems!

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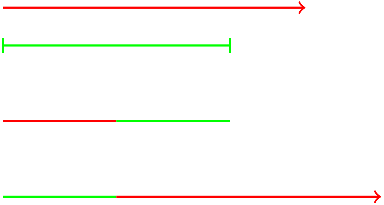

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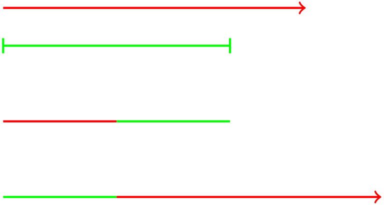

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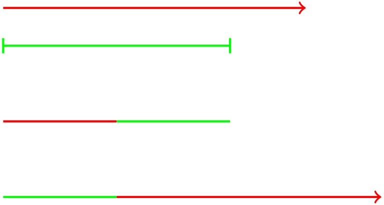

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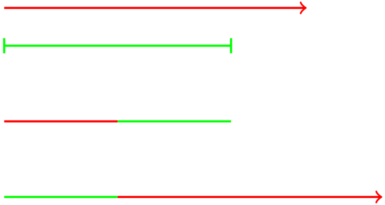

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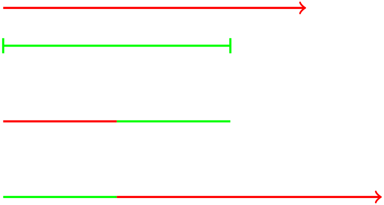

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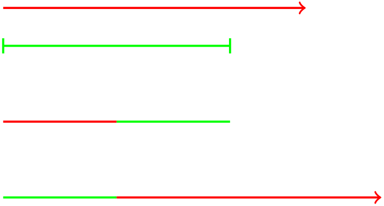

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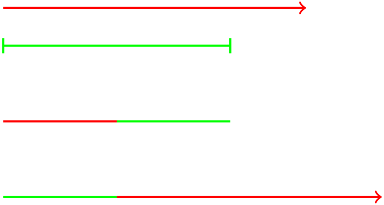

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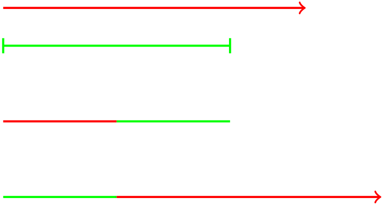

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$[0,1]$ is same cardinality as nonnegative reals!

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for a beautiful, succint, powerful idea.

What's the idea?

Rao is freaked out.

Are real numbers even real?

Almost all real numbers can't be described.

π?
The ratio of the perimeter of a circle to its diameter.

e? Transendental number.
$\lim_{n\to\infty}(1+1/n)^n$.

$\sqrt{2}$? Algebraic number.
A solution of $x^2 = 2$.

Really, rationals seem fine for... say... calculus.

$\lim_{n\to\infty}\sum_{i=0}^{n}\frac{(b-a)}{n}f(x_i)$, where $x_i = a + i\times(b-a)/n$.

So why real numbers?

$\int_a^b f(x)dx$ is beautiful, succint notation
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What's the idea? Area.

Generalized Continuum hypothesis.

There is no infinite set whose cardinality is between the cardinality of an infinite set and its power set.

Generalized Continuum hypothesis.

There is no infinite set whose cardinality is between the cardinality of an infinite set and its power set.

The powerset of a set is the set of all subsets.

Resolution of hypothesis?

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Gödel. 1940.
Can’t use math!

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Who shaves the barber?

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Uh oh....

The Barber!

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The barber shaves every person who does not shave themselves.

(A) Barber not Mark. Barber shaves Mark.
(B) Mark shaves the Barber.
(C) Barber doesn't shave themself.
(D) Barber shaves themself.

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Its all true.

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Its all true. It's all a problem.

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Recall: powerset of the naturals is not countable.

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