Probability

Precise model for Random Experiments

Randomness in computation
- randomness in data
- probabilistic algorithms
  e.g. Quick sort.

Probability: Precise, unambiguous way to argue about uncertainty

intuitive?

Example 1: Random Experiment – toss a fair coin

Possible Outcomes: H - Heads

T – Tails

Likelihoods: H – 50%

(aka probabilities) T – 50%

The Probability Model specifies:

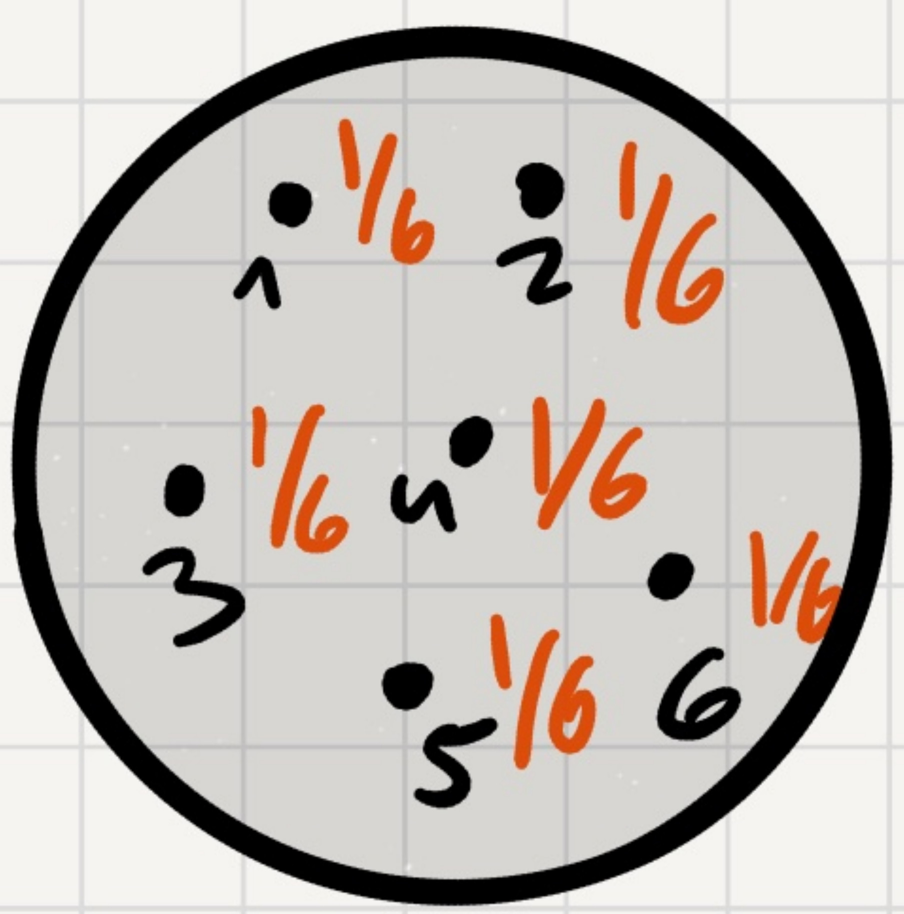
1. A set of possible outcomes $\Omega = \{H, T\}$.
2. A probability assigned to each outcome.

$$Pr[H] = Pr[T] = 0.5$$

Example 2: Random Experiment – toss a fair die

Sample Space $\Omega = \{1, 2, 3, 4, 5, 6\}$

Probabilities: $Pr[1] = 1/6$ $\quad Pr[2] = 1/6$ $\quad \dots \quad Pr[6] = 1/6$



Sample Space + Probabilities = Probability Space

Example 3:

Prob. of tossing heads is p

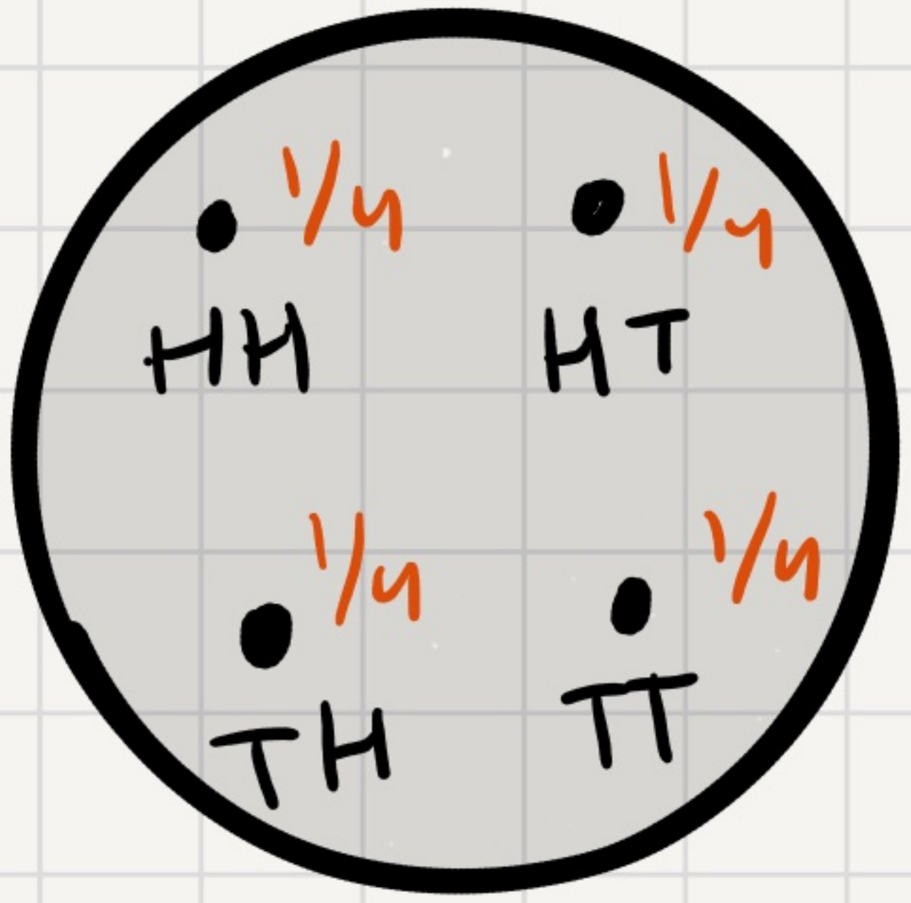
$\Pr[H] = p$ $\qquad\qquad$ $\Pr[T] = 1-p$

How can you estimate p?

Experiment 4: Flip two fair coins

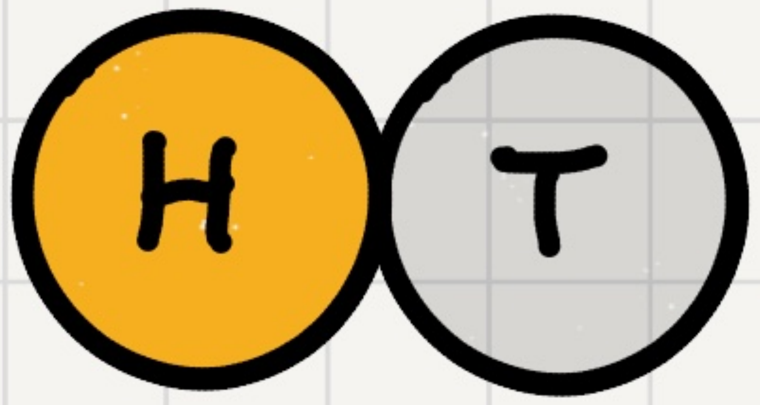
Possible Outcomes: $\{HH, HT, TH, TT\}$

Likelihoods/Probabilities: $1/4$ each.



Important: Sample Space is not $\{H, T\}$.

Experiment 5: Flip two glued coins



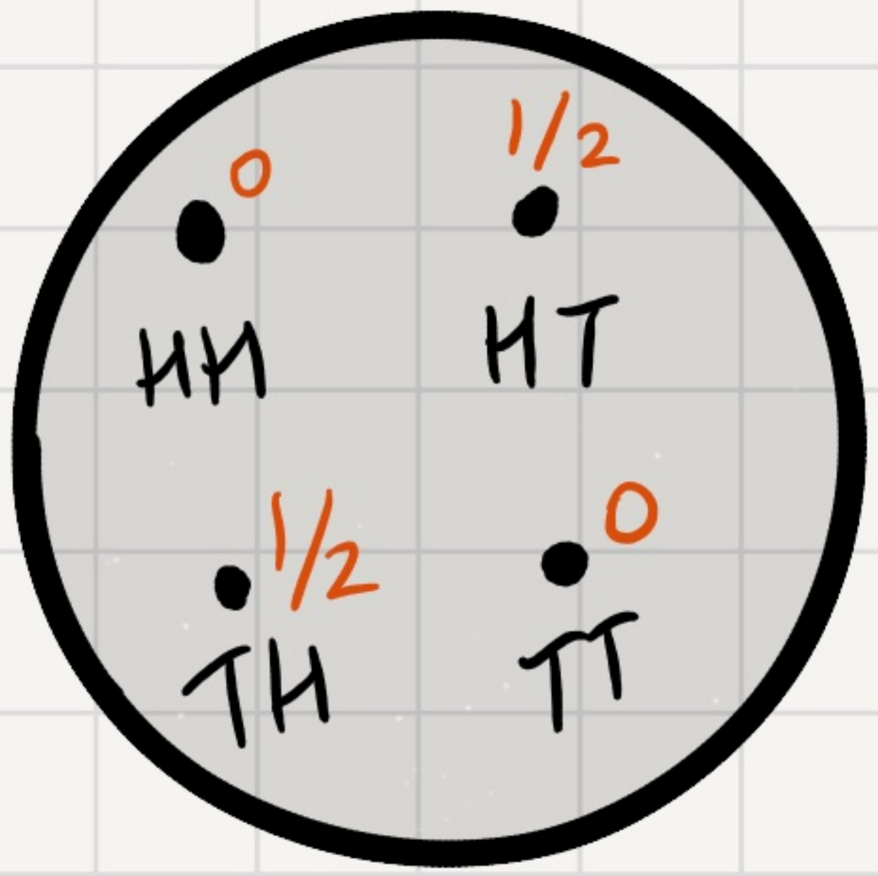
Possible Outcomes: $\{HH, HT, TH, TT\}$

Likelihoods / Probabilities: $Pr[HH] = 0$

$Pr[HT] = 1/2$

$Pr[TH] = 1/2$

$Pr[TT] = 0$



Experiment 6: Flip two biased coins with $pr[H]$ each $= 60\%$

$$\Omega = \{HH, HT, TH, TT\}$$

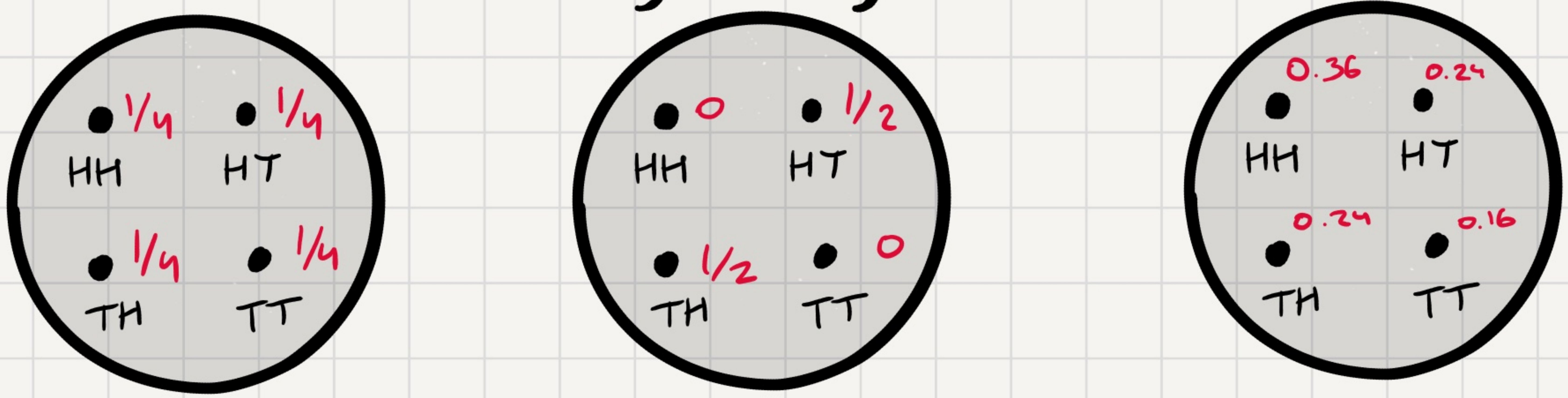
0.6×0.6 $\quad$ 0.6×0.4 $\quad$ 0.4×0.6 $\quad$ 0.4×0.4

Experiments 4 – 6

Two fair coins

Two coins glued together

Two biased coins



What can you say about these probabilities in general?

1. Non-negative
2. Sum up to 1.

Probability Space Formalism

Ω is the sample space (set of possible outcomes)

$\omega \in \Omega$ is a sample point (aka outcome)

Assign $P_r[\omega]$ to every $\omega \in \Omega$

- $\forall \omega \in \Omega \quad 0 \leq P_r[\omega] \leq 1$
- $\displaystyle\sum_{\omega \in \Omega} P_r[\omega] = 1.$

Uniform Probability Space

In a uniform prob. space each outcome is equally likely.

$$\Pr[\omega] = \frac{1}{|\Omega|} \quad \text{for all} \quad \omega \in \Omega.$$

Examples:

- Rolling a die
- Flipping two fair coins
- Dealing a poker hand.

Bag of Marbles

Prob of picking each marble is 1/6.

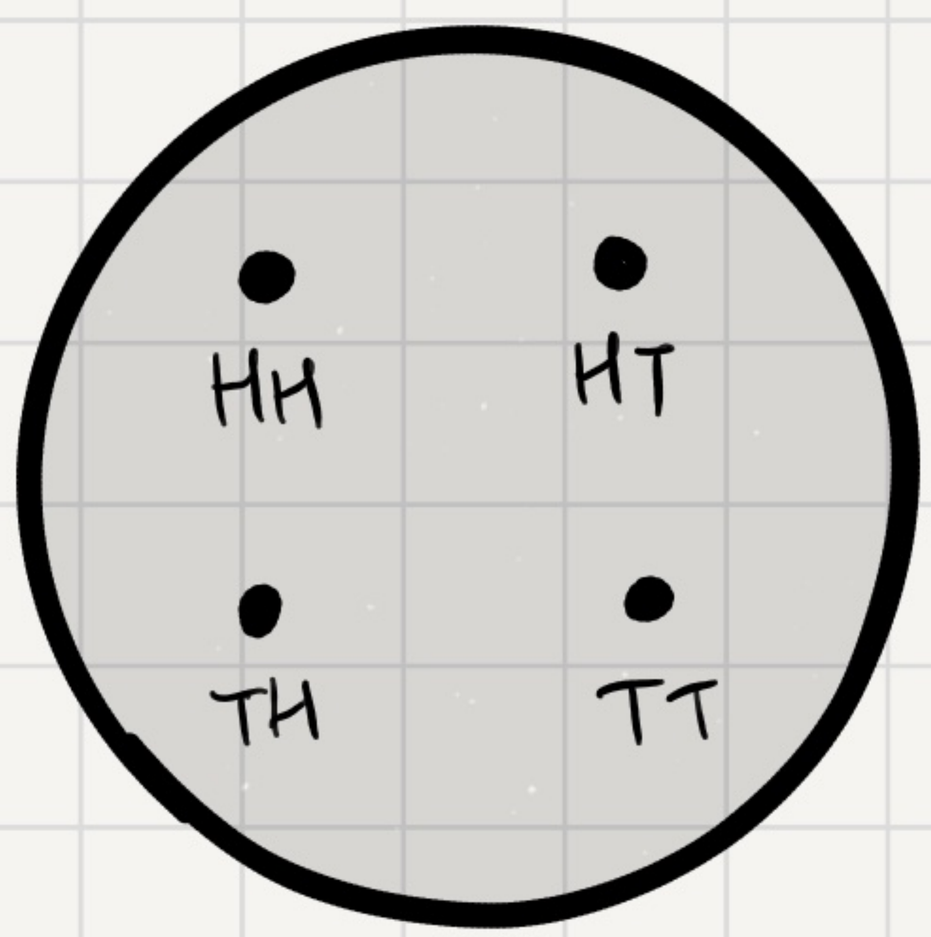
Questions about Random Experiments

We flip two fair coins.

What's the prob. of getting exactly one head?

$$\Omega = \{HH, HT, TH, TT\}$$

Intuitive Answer: $\Pr[HT] + \Pr[TH] = 1/4 + 1/4 = 1/2$.



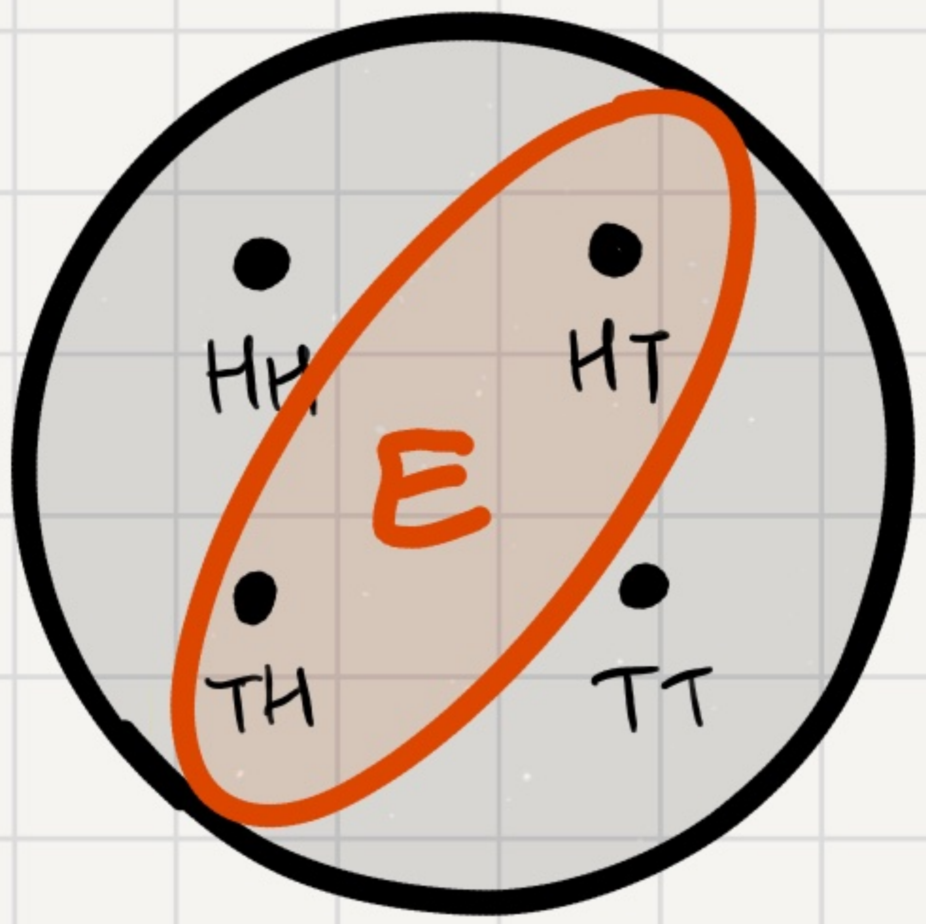
Events

- An event, E, is a subset of the possible outcomes:

$$E \subseteq \Omega$$

- The probability of E is defined as

$$\Pr[E] = \boxed{}$$



Uniform Probability Space

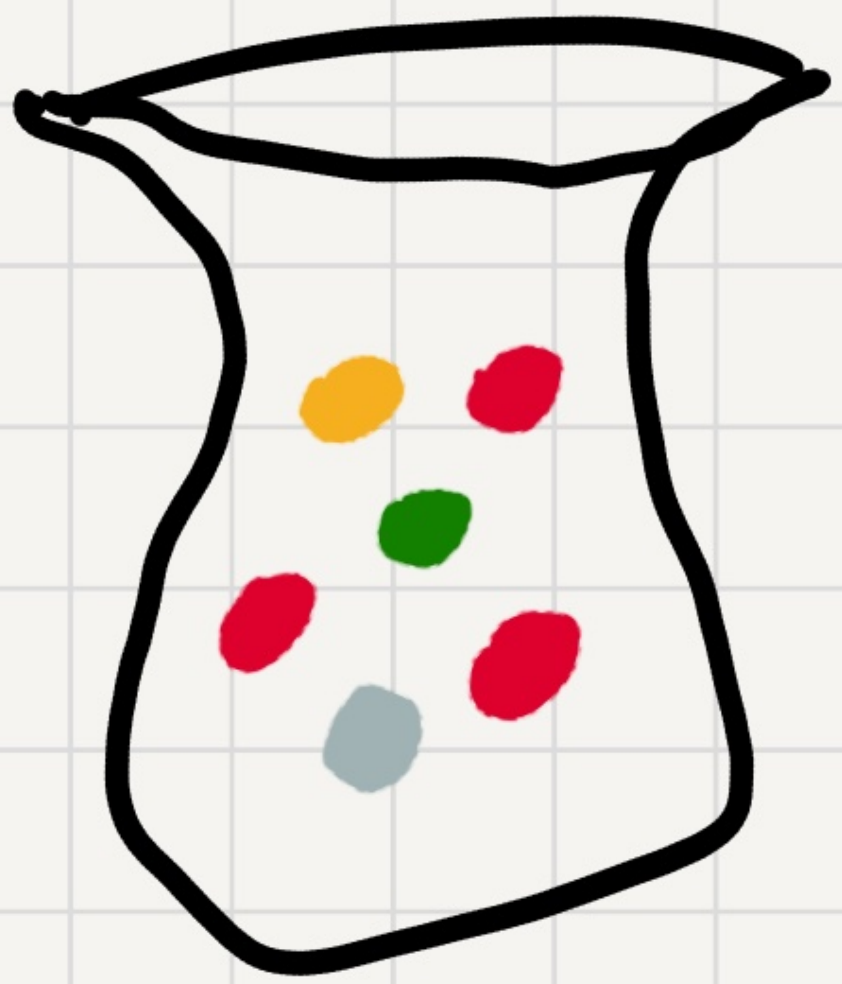
In a uniform prob. space each outcome is equally likely.

$$\Pr[\omega] = \frac{1}{|\Omega|} \quad \text{for all} \quad \omega \in \Omega.$$

Observation: In a uniform prob. space

$$\Pr[E] = \sum_{\omega \in E} \Pr[\omega] = \frac{|E|}{|\Omega|}$$

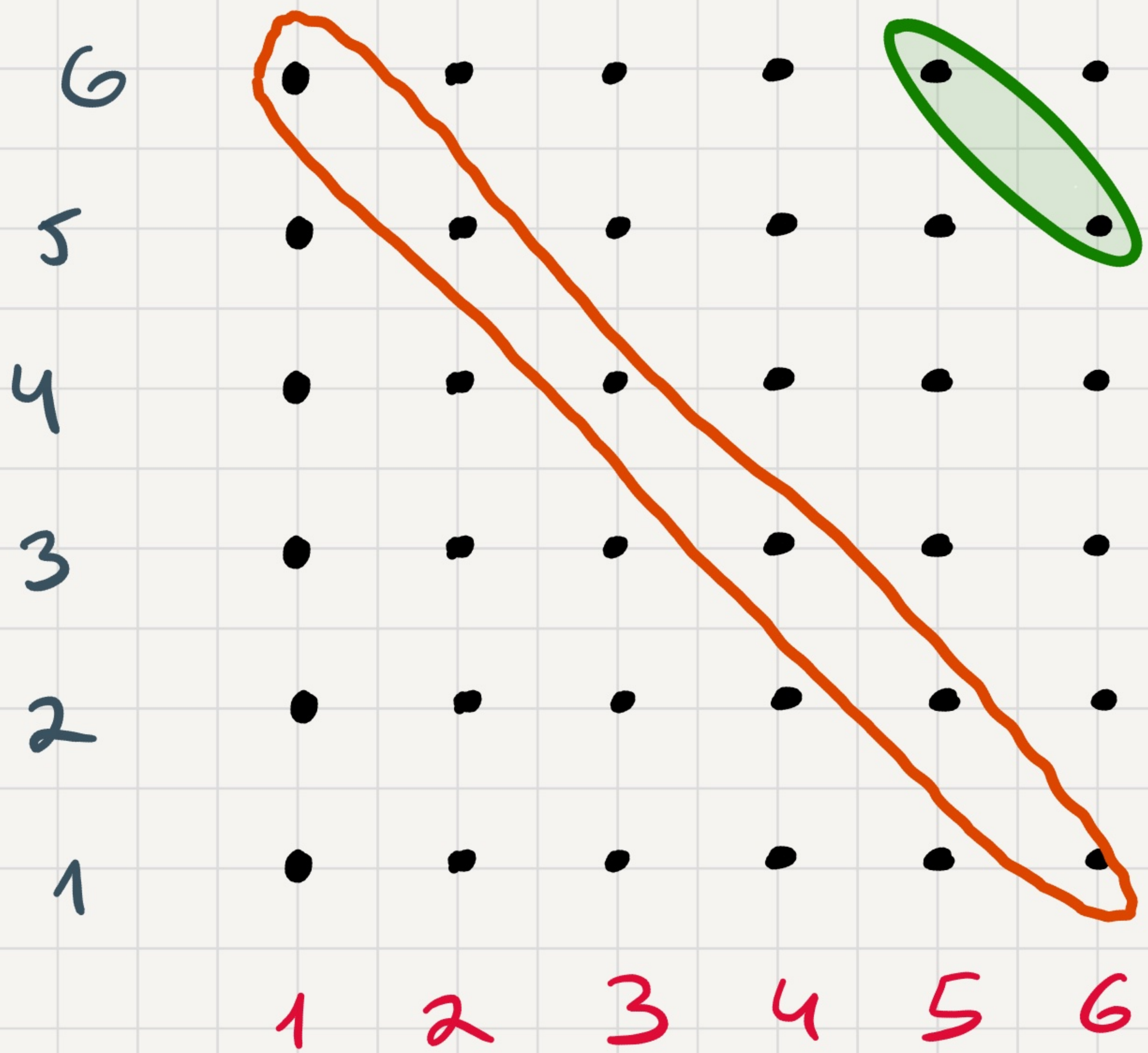
Bag of Marbles

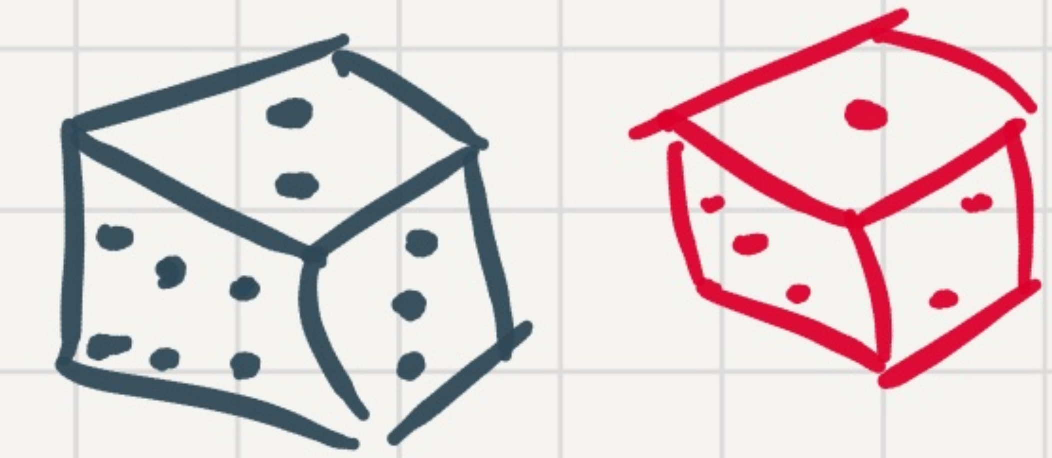


What's the prob. of picking a red marble?

Rolling two dice

What's the prob that the sum = 7?





Pr. of each outcome $= 1/36$.

E_1 = "sum to 7"

$Pr[E_1] =$

E_2 = "sum to 11"

$Pr[E_2] =$

Poll:

Random Experiment: Flip a fair coin 20 times

$$\Omega = \{HH\ldots H,\ HH\ldots HT,\ \ldots,\ TT\ldots T\}$$

Which outcome is more likely:

$\omega_1 =$ HHHHHHHHHHHHHHHHHHHH

$\omega_2 =$ HTHHTTHTTHHTHTTTHHTH

Which event is more likely:

E_1: 20 heads out of 20

E_2: 10 heads out of 20.

Poll:

Random Experiment: Flip a fair coin 20 times

$\Omega = \{HH\ldots H, HH\ldots HT, \ldots, TT\ldots T\}$ $|\Omega| = 2^{20}$

Which outcome is more likely:

$\omega_1 =$ HHHHHHHHHHHHHHHHHHHH

$\omega_2 =$ HTHHTTHTTHHTHTTTHHTH

Which event is more likely:

E_1: 20 heads out of 20 $\Pr[E_1] = \frac{1}{2^{20}}$

E_2: 10 heads out of 20.

$$\Pr[E_2] = \frac{|E_2|}{2^{20}} = \frac{\binom{20}{10}}{2^{20}} = \frac{184{,}756}{2^{20}}$$

Flip 20 fair coins

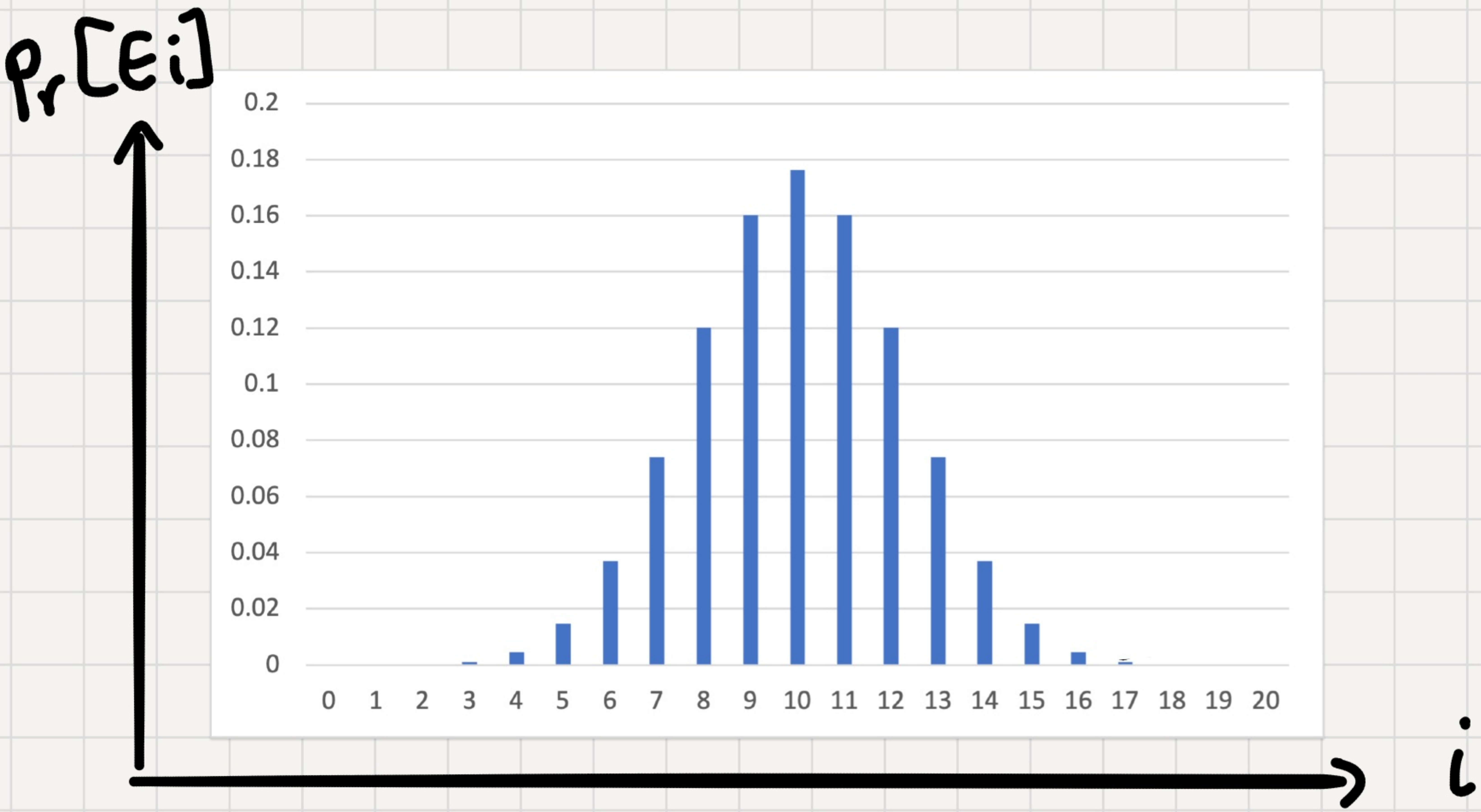
$E_0 =$ 0 heads out of 20

$E_1 =$ 1 heads out of 20

$\vdots$

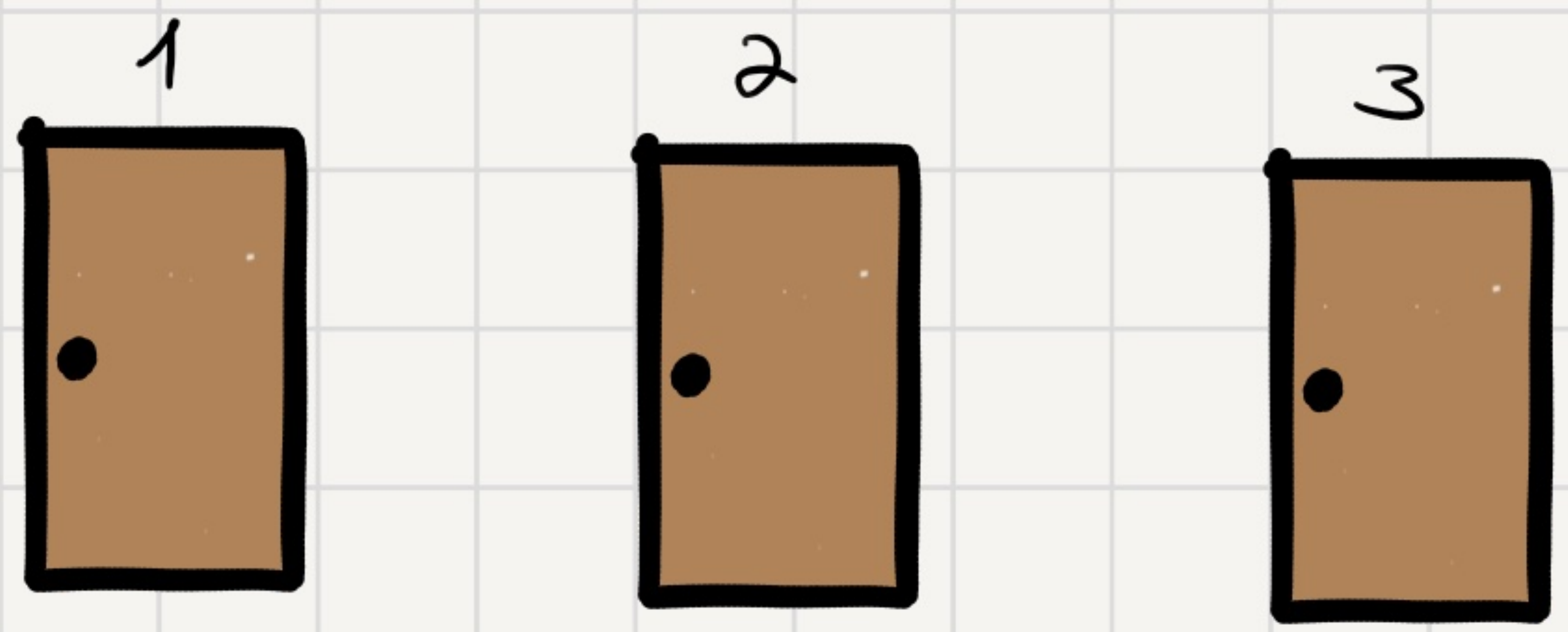
$E_{20} =$ 20 heads out of 20

$\left.\right\}$ 21 different events.



Random Experiment (Somewhat Confusing)

Monty Hall: Gameshow



3 doors

1 prize = car

2 goats

1. The host places the prize behind a randomly selected door.
2. You initially pick door # 1.
3. The host, who knows where the prize is, opens one of the doors $\{2,3\}$ that has a goat.
4. The host offers you to switch doors. Should you?

$$\Omega = \{ \underset{\text{prize location}}{(1}, \underset{\text{door opened by host}}{2)}, (1,3), (2,3), (3,2) \}$$

$$\Pr[(1,2)] = \frac{1}{3} \cdot \frac{1}{2}$$

$$\Pr[(1,3)] = \frac{1}{3} \cdot \frac{1}{2}$$

$$\Pr[(2,3)] = \frac{1}{3}$$

$$\Pr[(3,2)] = \frac{1}{3}$$

"Always stay" strategy

$\Pr[\text{winning in always stay}]$

$= \Pr[(1,2)] + \Pr[(1,3)] = 1/3$

"Always switch" strategy

$\Pr[\text{winning in always switch}]$

$= \Pr[(2,3)] + \Pr[(3,2)]$

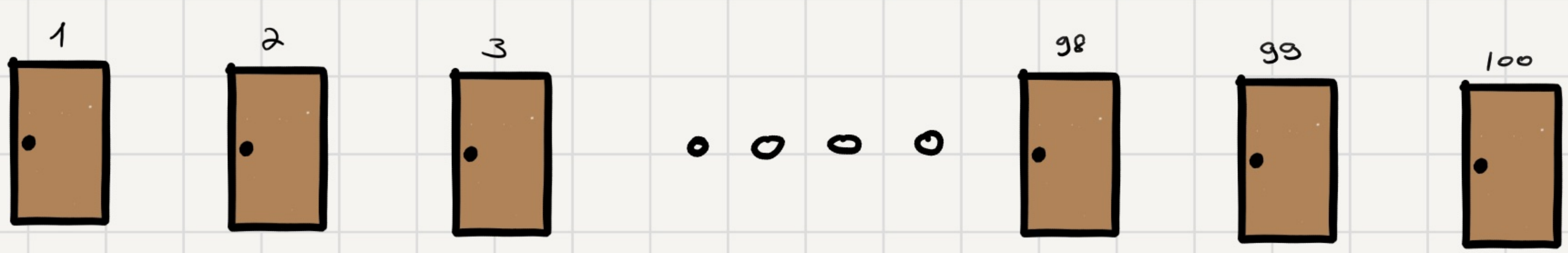
$= 1/3 + 1/3 = 2/3.$

Let's demonstrate the game with 100 doors.

I allow you to make the first choice.

I open all the doors but ____

Would you like to switch?



Summary

We model random experiment as

Probability Space $\begin{cases} \text{• Sample Space } \Omega \text{ – set of possible outcomes} \\ \text{• Probabilities assigned to each outcome } \omega \in \Omega. \end{cases}$

- $\forall \omega \in \Omega \qquad 0 \leq Pr[\omega] \leq 1$
- add up to 1: $\sum_{\omega \in \Omega} Pr[\omega] = 1$

An Event E is a subset of Ω.

$$Pr[E] = \sum_{\omega \in E} Pr[\omega].$$

Uniform probability space $\forall \omega \in \Omega$: $Pr[\omega] = \frac{1}{|\Omega|}$

$$Pr[E] = \frac{|E|}{|\Omega|}$$