Lecture 17

Bayes Rule & Applications in CS.

Lecture 16 Summary

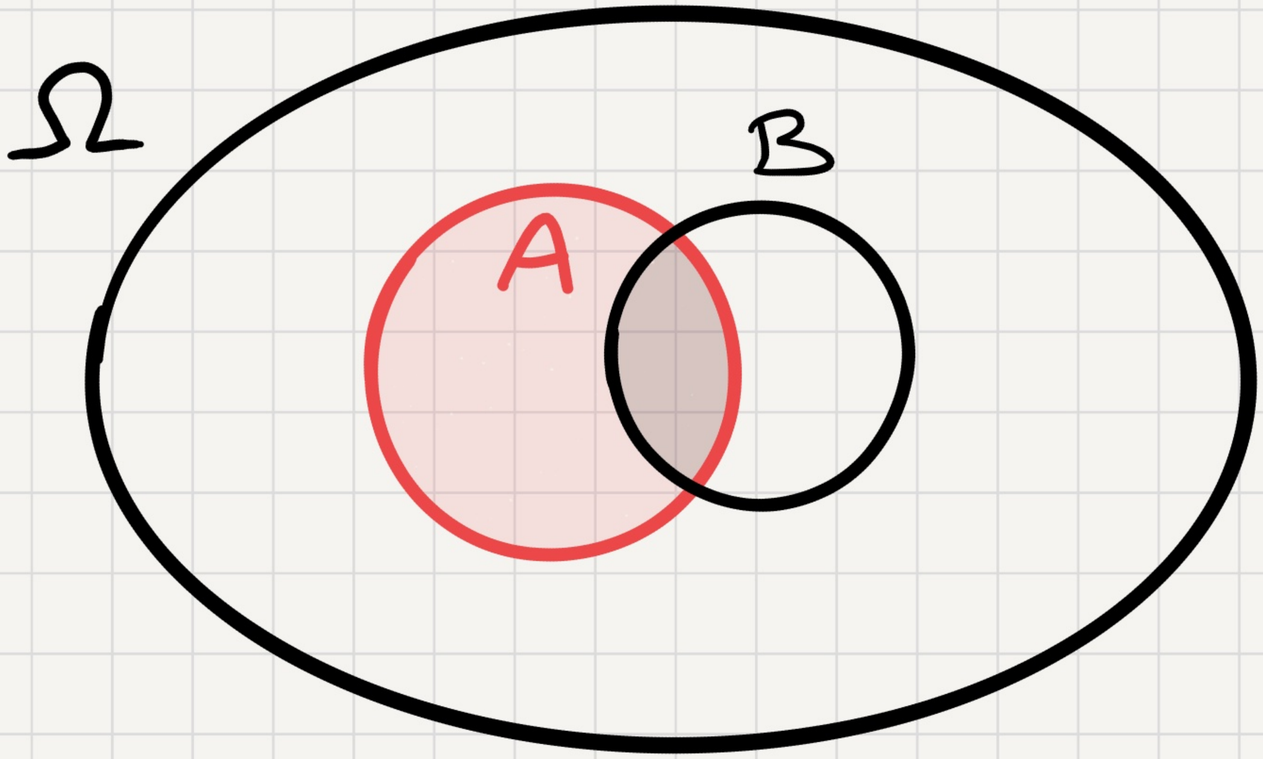
- Probability is additive
- Union Bound $\Pr[A_1 \cup \dots \cup A_n] \leq \Pr[A_1] + \dots + \Pr[A_n]$
- Inclusion-Exclusion $\Pr[A \cup B] = \Pr[A] + \Pr[B] - \Pr[A \cap B]$
- Total probability: if $A_1, \dots, A_n$ partition Ω

  then $\Pr[B] = \Pr[A_1 \cap B] + \dots + \Pr[A_n \cap B]$
- Conditional probability: $\Pr[B|A] = \Pr\frac{[A \cap B]}{\Pr[A]}$
- Independence: $\Pr[A \cap B] = \Pr[A] \cdot \Pr[B]$
- Correlation $\Pr[A \cap B] > \Pr[A] \cdot \Pr[B]$
- Bayes Rule.

Conditional Probability

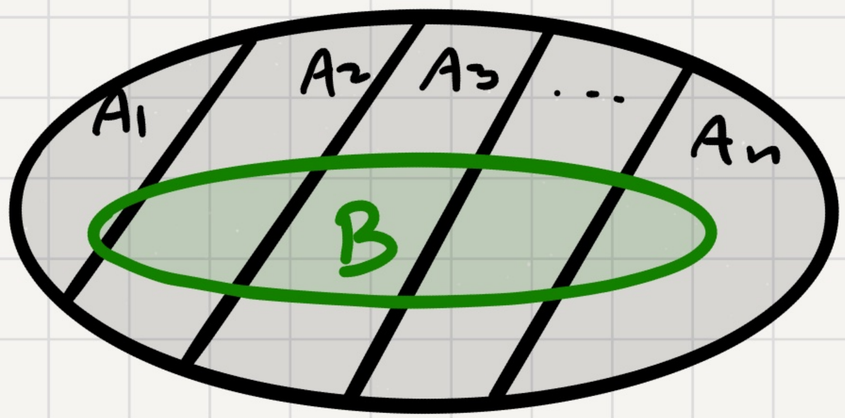
Defn: The conditional probability of event B given event A is

$$\Pr[B \mid A] = \frac{\Pr[A \cap B]}{\Pr[A]}$$



Total Probability with Conditional Probability

Assume Ω is a union of disjoint events $A_1, \ldots, A_n$.



Since B is the disjoint union of $B \cap A_1, \ldots, B \cap A_n$

$$Pr[B] = Pr[B \cap A] + \ldots + Pr[B \cap A_n]$$

Total Probability with Conditional Probability

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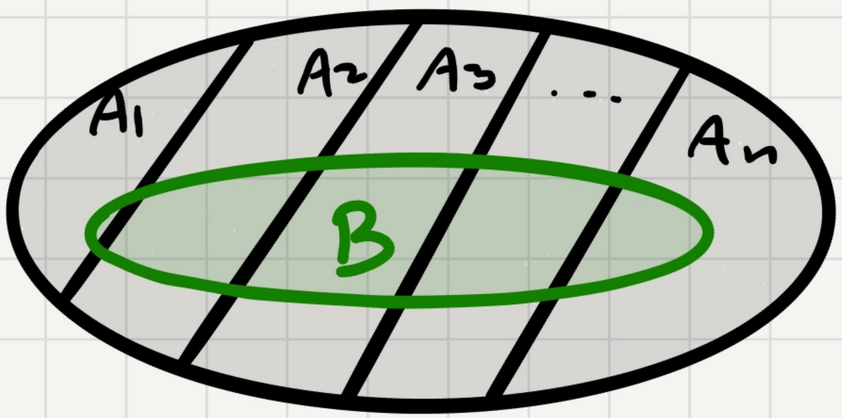


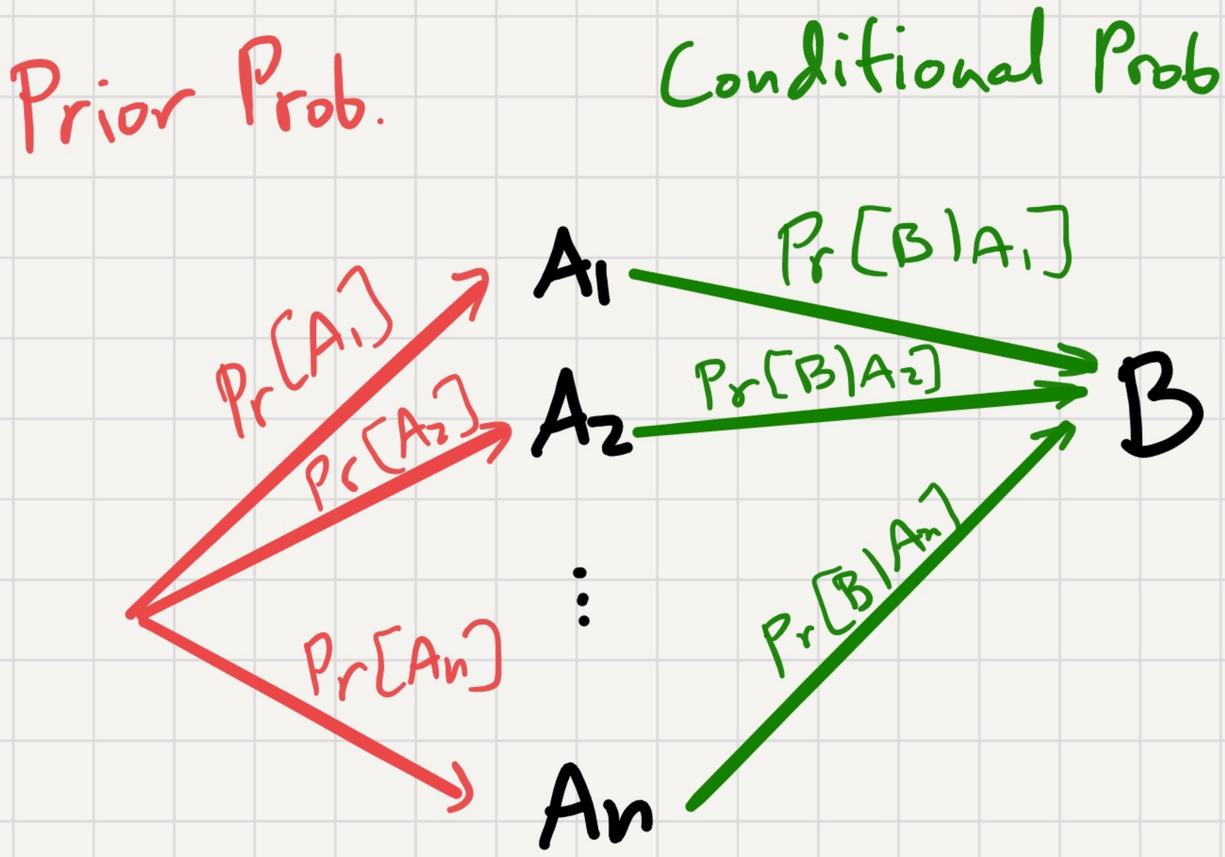
Since B is the disjoint union of $B \cap A_1, \ldots, B \cap A_n$

$$Pr[B] = Pr[B \cap A_1] + \ldots + Pr[B \cap A_n]$$

Thus, by the product rule

$$Pr[B] = Pr[A_1] \cdot Pr[B \mid A_1] + \ldots + Pr[A_n] \cdot Pr[B \mid A_n]$$

Total Probability Rule with Conditional Probability



$$Pr[B] = Pr[A_1] \cdot Pr[B|A_1] + \ldots + Pr[A_n] \cdot Pr[B|A_n]$$

Bayes Rule

Suppose you know $Pr[B|A], Pr[A], Pr[B]$

What's $Pr[A|B]$?

$$Pr[A|B] = \frac{Pr[A \cap B]}{Pr[B]} = \frac{Pr[A] \cdot Pr[B|A]}{Pr[B]}$$

Bayes Rule Example #1

Experiment:

1. Pick at random either a fair coin
   or a biased coin with 60% chance heads.
2. Toss the coin you picked

$\Omega = \{(\text{fair}, H), (\text{fair}, T), (\text{biased}, H), (\text{biased}, T)\}$

A = "coin is fair" B = "got H"

WTK: $Pr[A|B]$

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WTK: $\Pr[A|B] = \frac{\Pr[A \cap B]}{\Pr[B]} = \frac{\Pr[A] \cdot \Pr[B|A]}{\Pr[B]} = \frac{0.5 \times 0.5}{\Pr[B]}$

What's $\Pr[B]$?

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$$Pr[B] = Pr[A \cap B] + Pr[\bar{A} \cap B]$$
$$= Pr[A] Pr[B|A] + Pr[\bar{A}] \cdot Pr[B|\bar{A}]$$

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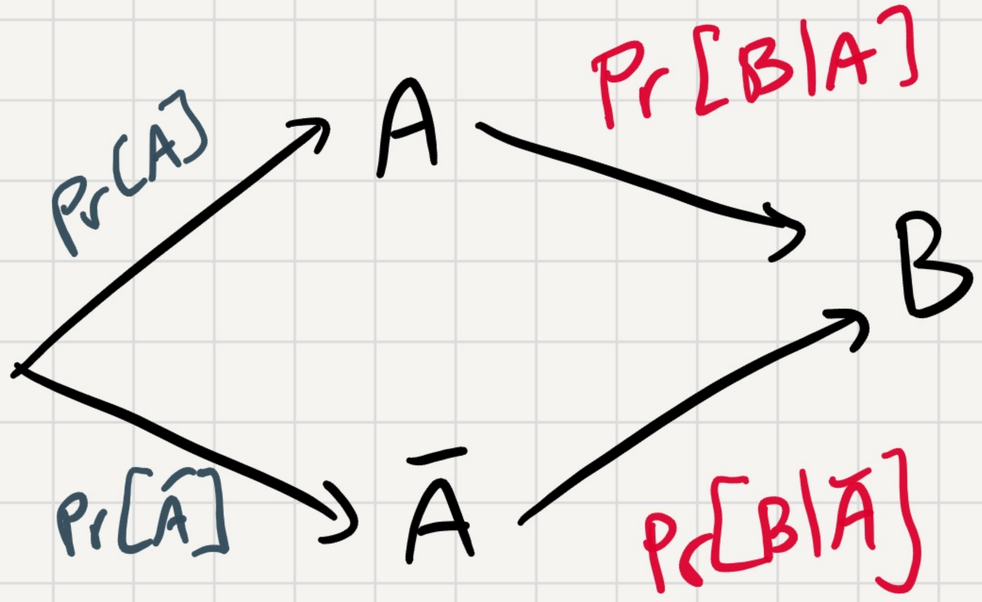
WTK: $$Pr[A|B] = \frac{Pr[A \cap B]}{Pr[B]} = \frac{Pr[A] \cdot Pr[B|A]}{Pr[B]} = \frac{0.25}{Pr[B]}$$

$$Pr[B] = Pr[A]Pr[B|A] + Pr[\bar{A}] \cdot Pr[B|\bar{A}]$$
$$= 0.5 \times 0.5 + 0.5 \times 0.6 = 0.55$$



$$Pr[B] = Pr[A] \cdot Pr[B|A] + Pr[\bar{A}] \cdot Pr[B|\bar{A}]$$

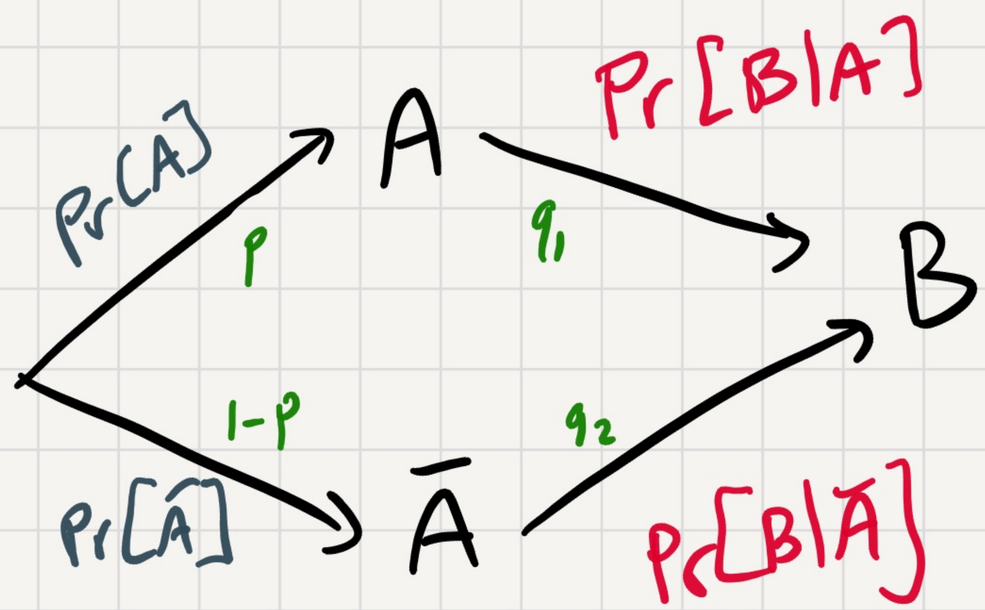
Bayes Rule (updated)

$$Pr[A|B] = \frac{Pr[A \wedge B]}{Pr[B]}$$

$$= \frac{Pr[A] \cdot Pr[B|A]}{Pr[A] \cdot Pr[B|A] + Pr[\bar{A}] \cdot Pr[B|\bar{A}]}$$

Bayes

$$Pr[A|B] = \frac{Pr[A]\,Pr[B|A]}{Pr[B]}$$



$$Pr[B] = Pr[A]\cdot Pr[B|A] + Pr[\bar{A}]\cdot Pr[B|\bar{A}]$$

Bayes Rule (updated)

$$Pr[A|B] = \frac{Pr[A]\cdot Pr[B|A]}{Pr[A]\cdot Pr[B|A] + Pr[\bar{A}]\cdot Pr[B|\bar{A}]} = \frac{p\,q_1}{p\,q_1 + (1-p)\,q_2}$$

Bayes Rule Example #2

Suppose there's a disease that occur in 0.001 of the population.

There's a test for the disease.

For a random person: $\Pr[\text{test positive} \mid \text{sick}] = 0.99$

$\Pr[\text{test positive} \mid \text{not sick}] = 0.01$

A random person arrives and tests positive

Q: what's the likelihood that he has the disease?

$Pr[sick] = 0.001$

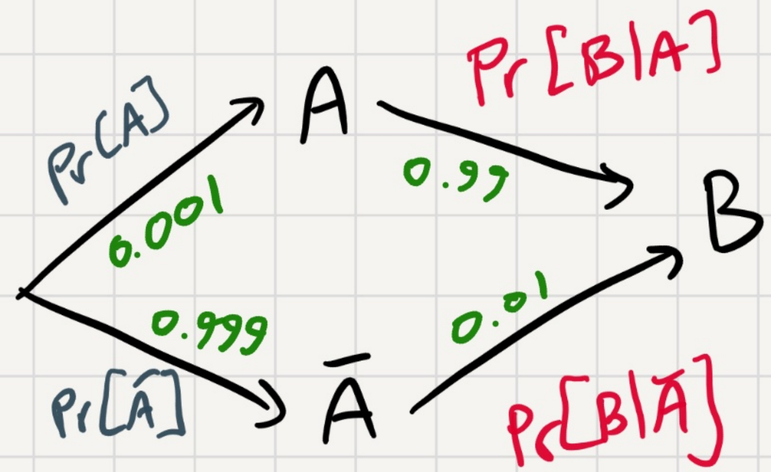
$Pr[positive \mid sick] = 0.99$

$Pr[positive \mid not\ sick] = 0.01$

$A =$ "sick"

$B =$ "tested positive"

WTK: $Pr[sick \mid positive]$



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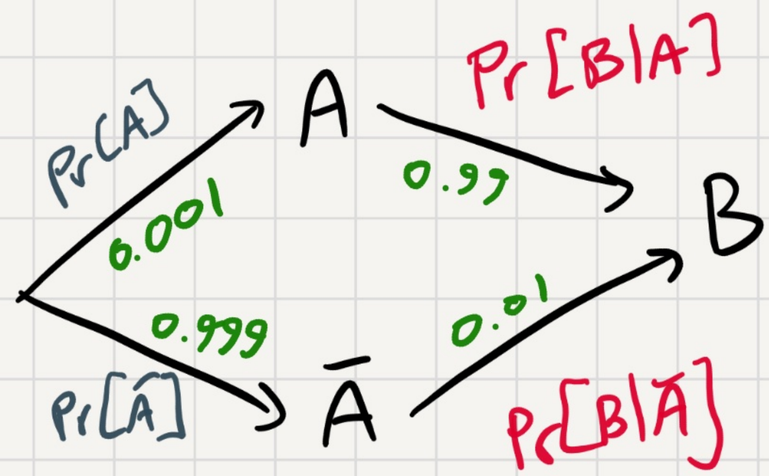

$$Pr[A \mid B] = \frac{Pr[A \cap B]}{Pr[B]}$$

$$= \frac{Pr[A] \cdot Pr[B \mid A]}{Pr[A] \cdot Pr[B \mid A] + Pr[\bar{A}]\, Pr[B \mid \bar{A}]} =$$

$Pr[sick] = 0.001$

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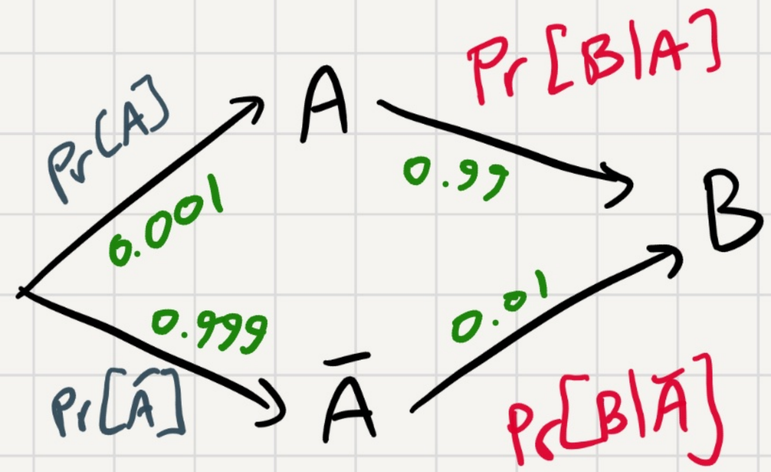
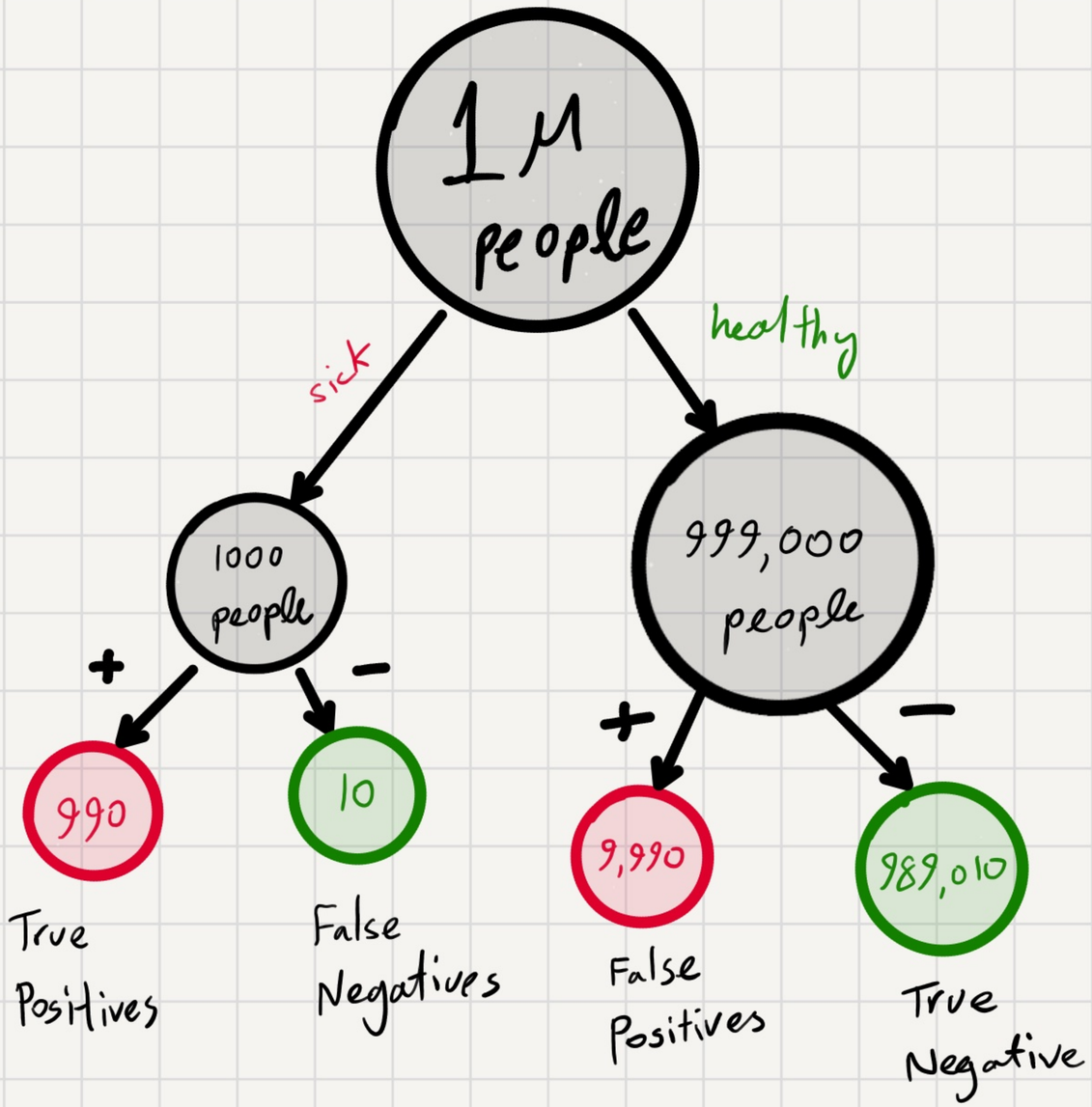


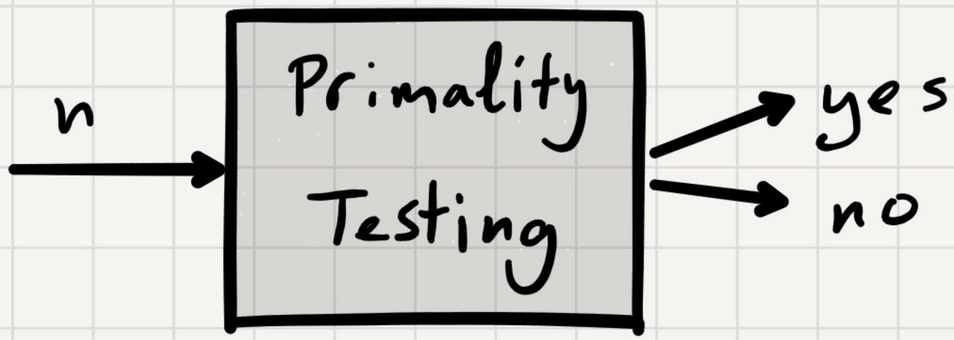
$$Pr[A \mid B] = \frac{Pr[A \cap B]}{Pr[B]}$$

$$= \frac{Pr[A] \cdot Pr[B \mid A]}{Pr[A] \cdot Pr[B \mid A] + Pr[\bar{A}]\, Pr[B \mid \bar{A}]} = \frac{0.001 \times 0.99}{0.001 \times 0.99 + 0.999 \times 0.01}$$

$$\approx 0.09$$


1M people
sick
healthy
1000 people
999,000 people
+
−
+
−
990
10
9,990
989,010
True Positives
False Negatives
False Positives
True Negative

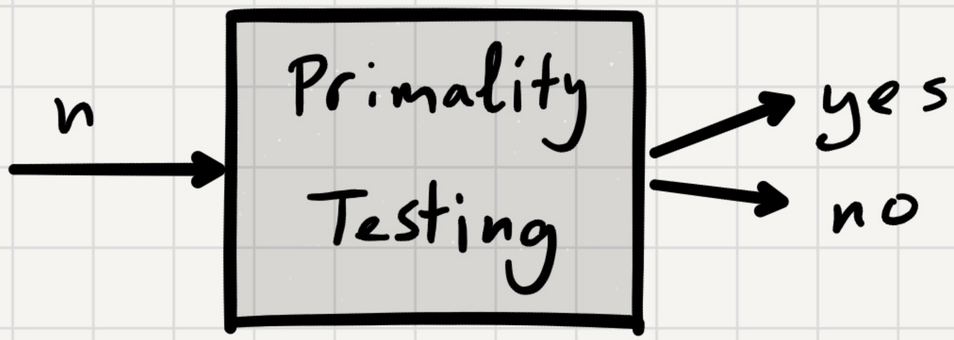
Primality Testing



Properties:

- If n is prime, always output "yes"
- If n is composite, outputs "no" w.p. $1 - \frac{1}{1000}$

Primality Testing



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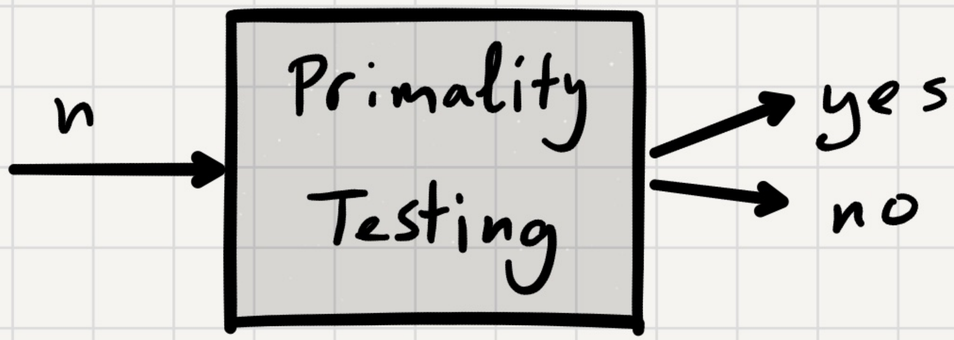
what's $\Pr[n \text{ is prime} \mid \text{output is "yes"}]$?

Primality Testing



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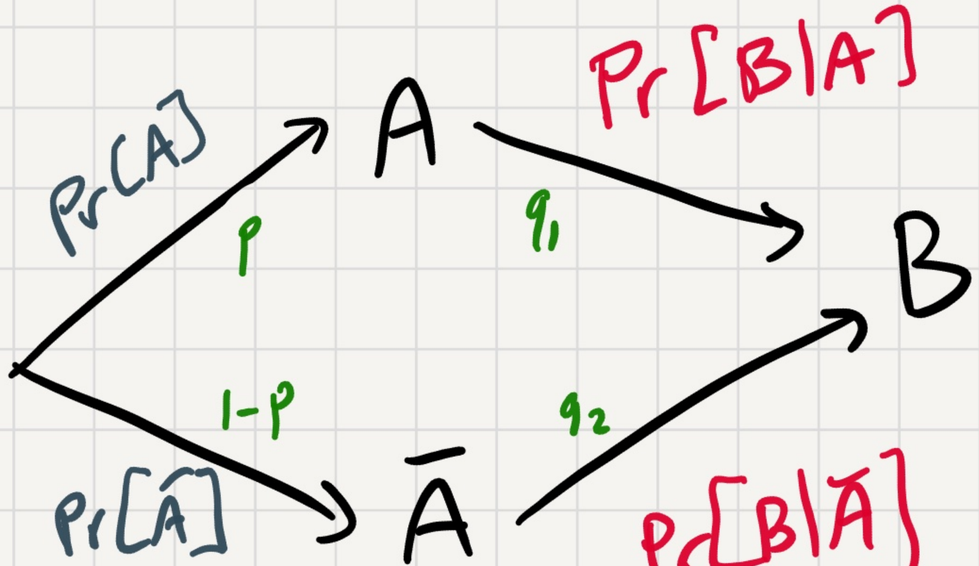
what's $\Pr[n \text{ is prime} \mid \text{output is "yes"}]$?

For that, n should be random as well.

- Pick $n \in [2^{511}, 2^{512})$ uniformly at random.

Primality Testing



$$Pr[A|B] = \frac{p\,q_1}{p\,q_1 + (1-p)\,q_2}$$

A = "n is prime" B = "answer is yes"

$$p = \frac{1}{350} \qquad q_1 = 1 \qquad q_2 = \frac{1}{1000}$$

$$Pr[A|B] = \frac{p \cdot q_1}{p \cdot q_1 + (1-p) \cdot q_2} = \frac{\frac{1}{350}}{\frac{1}{350} + \frac{349}{350} \cdot \frac{1}{1000}} \approx 0.74$$

Birthday Paradox

Poll: How many students do you need to have in a classroom so that you'll **definitely** have a pair of students with the same birthday?

A: 23

B: 365

C: 366

D: 365×364

Birthday Paradox

How many students do you need to have in a classroom so that you'll have a pair of students with the same birthday w.p. $\geq \frac{1}{2}$.

Birthday Paradox

Assumption: Each student has a uniformly random birthday out of the 365 options.

---

k students.

$\Omega = \{(x_1, \ldots, x_k) : \forall i \quad 1 \le x_i \le 365\}$ $|\Omega| =$

$E =$ "no collision" $= \{(x_1, \ldots, x_k) : \text{For all } i<j,\ x_i \neq x_j\}$

$|E| =$

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$$\Omega = \{(x_1, \ldots, x_k) : \forall i \quad 1 \le x_i \le 365\} \qquad |\Omega| = 365^k$$

$$E = \text{"no collision"} = \{(x_1, \ldots, x_k) : \text{For all } i<j,\ x_i \neq x_j\}$$

$$|E| = 365 \cdot 364 \cdots (365-k+1)$$

$$Pr[E] = \frac{|E|}{|\Omega|} = \frac{365 \cdot \cdots (365-k+1)}{365^k}$$

By tedious calculation when $k=23$, $Pr[E] \le 0.5$

and hence $Pr[\text{exists a collision}] \ge 0.5$

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$\overline{E}$ = "$\exists$ collision" = $\{(x_1, \ldots, x_k) : \exists\ i<j \text{ s.t. } x_i = x_j\}$

How to get a simple upper bound on $\Pr[\overline{E}]$?

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How to get a simple upper bound on $\Pr[\bar{E}]$?

For $1 \le i < j \le k$ let $A_{i,j}$ be the event indicating "$x_i = x_j$"

$$\Pr[\bar{E}] = \Pr\left[\bigcup_{1 \le i < j \le k} A_{i,j}\right]$$

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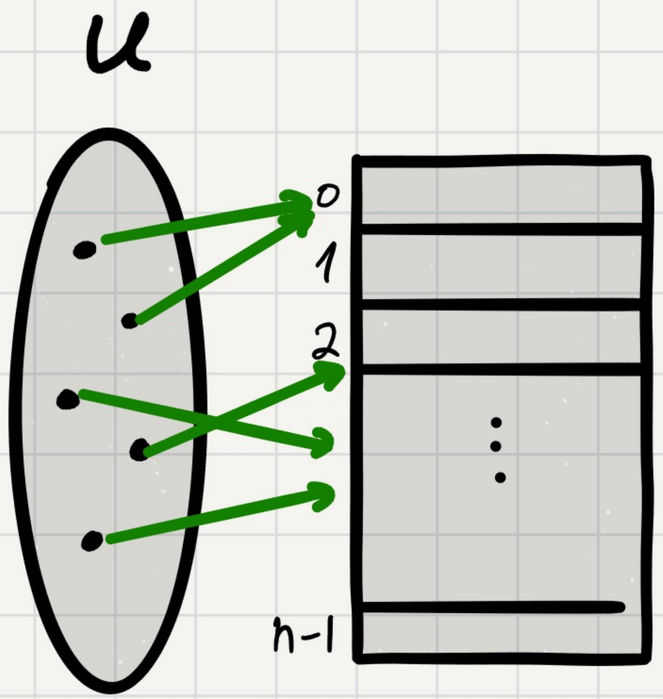
$$\Pr[\bar{E}] = \Pr\left[\bigcup_{1 \le i < j \le k} A_{i,j}\right] \le \sum_{1 \le i < j \le k} \Pr[A_{i,j}] = \binom{k}{2} \cdot \frac{1}{365}.$$

Birthday Paradox in Computing

You create a hash table of size n and add k elements to it using a random hash function.



Q: what's the prob. for a collision?

Birthday Paradox in Computing

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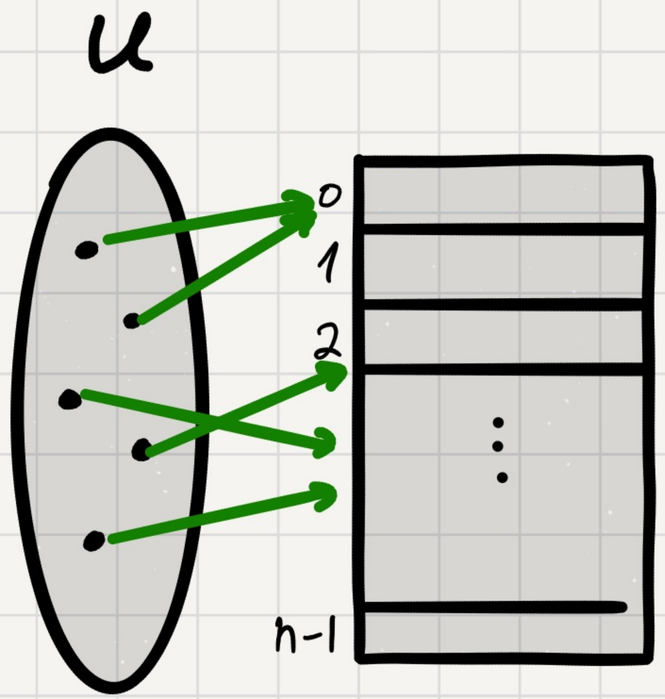

Q: what's the prob. for a collision?

$$Pr[\text{no collision}] = \frac{n \cdots (n-k+1)}{n^k}$$

$$Pr[\exists \text{ collision}] \leq \sum_{1 \leq i < j \leq k} Pr[\text{elements } i \& j \text{ collide}] = \binom{k}{2} \cdot \frac{1}{n}.$$

n balls in n bins

What's the maximum capacity?

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What's the maximum capacity?

Let j be a parameter.

Denote by A_j = "there exists a bin with $\geq j$ balls"

$Pr[A_j] = ?$

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Let j be a parameter.

Denote by A_j = "there exists a bin with $\geq j$ balls"

$\Pr[A_j] = ?$

Let $A_{j,1}, \ldots, A_{j,n}$ be events $A_{j,i}$ = "bin #i has $\geq j$ balls"

$A_j = A_{j,1} \cup \ldots \cup A_{j,n}$ $\quad \Pr[A_j] \leq \sum_{i=1}^{n} \Pr[A_{j,i}]$

n balls in n bins

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$\Pr[A_{j,i}] \leq$

n balls in n bins

What's the maximum capacity?

$A_{j,i}$ = "bin # i has $\geq j$ balls" $\qquad$ A_j = "$\exists$ bin with $\geq j$ balls"

$$\Pr[A_{j,i}] \leq \frac{\binom{n}{j} \cdot n^{n-j}}{n^n} = \binom{n}{j} \cdot \frac{1}{n^j} = \frac{n \cdot \ldots \cdot (n-j+1)}{j!\, n^j} \leq \frac{1}{j!}$$

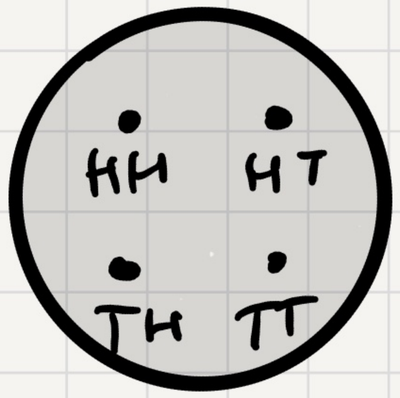
$$\Pr[A_j] \leq \Pr[A_{j,1}] + \ldots + \Pr[A_{j,n}] \leq n \cdot \frac{1}{j!}$$

For example for $n = 1{,}000{,}000$, $j = 10$, $\Pr[A_j] \leq 0.28$

Independence

We say that events A and B are independent if $\Pr[A \cap B] = \Pr[A] \cdot \Pr[B]$.

Example: Toss two fair coins



A = "first coin is H"

B = "second coin is H"

C = "same result"

Poll: Which are independent

1: A, B

2: A, C

3: B, C

Independence

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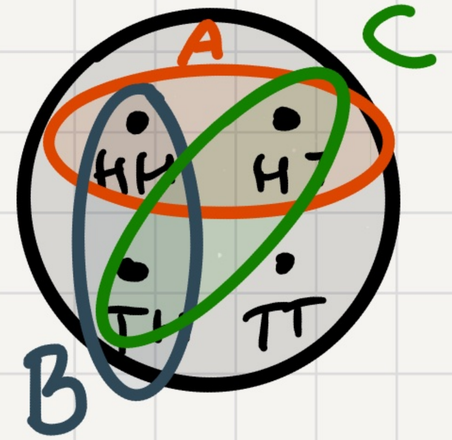

A & B are "indep.

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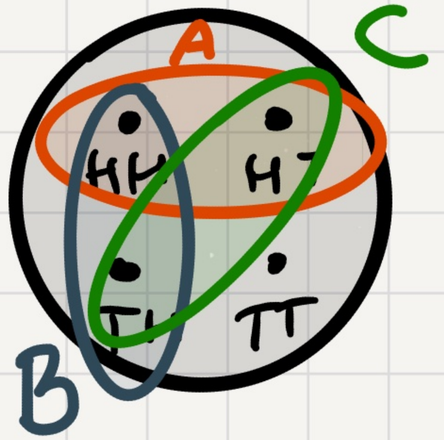

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Are A, B, C independent together?

Independence

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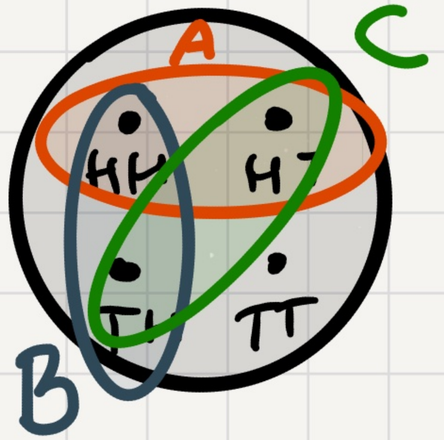
Example: Toss two fair coins



$A =$ "first coin is H"

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Are A, B, C independent together?

$\Pr[C \mid A \cap B] = 1$

Definition Pairwise Independence

We say that events $A_1, \ldots, A_n$ are pairwise independent if

$\forall\, i \neq j$ A_i and A_j are independent.

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Definition Mutual Independence

We say that events A_1, A_2, A_3 are mutually independent if they are pairwise indep. and $Pr[A_1 \cap A_2 \cap A_3] = Pr[A_1] \cdot Pr[A_2] \cdot Pr[A_3]$

Definition Pairwise Independence

We say that events $A_1, \ldots, A_n$ are pairwise independent if

$\forall\, i \neq j$ A_i and A_j are independent.

Definition Mutual Independence

We say that events $A_1, \ldots, A_n$ are mutually independent if for each non-empty subset $I \subseteq \{1, \ldots, n\}$ $\Pr[\bigcap_{i \in I} A_i] = \prod_{i \in I} \Pr[A_i]$

Secret Sharing

n shares threshold: 3

Degree 2 polynomial

How to pick the polynomial: Let $p > n$ be a prime

1. Pick $a_0 \in \{0, 1, \ldots, p-1\}$ uniformly at random.
2. Pick $a_1, a_2 \in \{0, 1, \ldots, p-1\}$ " " "

$$f(x) = a_2 x^2 + a_1 x + a_0$$

Share i: $f(i)$

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$$f(x) = a_2 x^2 + a_1 x + a_0$$

Share i: $f(i)$

$$Pr\left[s = a \mid f(x_1) = y_1, f(x_2) = y_2 \right] =$$

Secret Sharing

n shares threshold: 3

Degree 2 polynomial

How to pick the polynomial: Let $p > n$ be a prime

1. Pick $s = a_0 \in \{0, 1, \ldots, p-1\}$ uniformly at random.

2. Pick $a_1, a_2 \in \{0, 1, \ldots, p-1\}$ " " "

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Share i: $f(i)$

$$Pr[s = a \mid f(x_1) = y_1, f(x_2) = y_2] =$$

$$= \frac{Pr[s = a, f(x_1) = y_1, f(x_2) = y_2]}{Pr[f(x_1) = y_1, f(x_2) = y_2]} = \frac{1/p^3}{p/p^3} = 1/p$$

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Every three shares are independent:

$$\Pr[f(x_1)=y_1 \cap f(x_2)=y_2 \cap f(x_3)=y_3] = \frac{1}{p^3}$$

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but 4 shares are not.