Lecture 18

Random Variables I

Plan for Today:

- Random Variables
- Distributions
- Expectation

Numerical Questions about Random Experiments

Experiment: roll two dice

Q: What's their sum?

Numerical Questions about Random Experiments

Experiment: roll two dice

Q: What's their sum?

Experiment: toss a fair coin 100 times

Q: How many H you got?

Numerical Questions about Random Experiments

Experiment: roll two dice

Q: What's their sum?

Experiment: toss a fair coin 100 times

Q: How many H you got?

Experiment: Hand back assignments to 3 students at random

Q: How many students got their assignment back?

Numerical Questions about Random Experiments

Experiment: roll two dice

Q: What's their sum?

Experiment: toss a fair coin 100 times

Q: How many H you got?

Experiment: Hand back assignments to 3 students at random

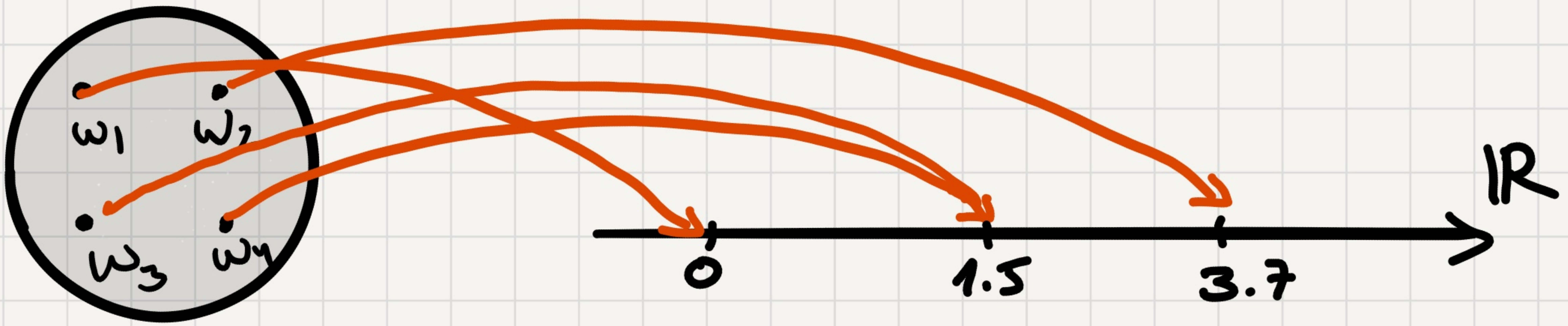
Q: How many students got their assignment back?

The answers to these questions are random variables.

Formal Definition

A **random variable**, X, for a random experiment with sample space Ω is a **function** mapping sample points to real numbers.

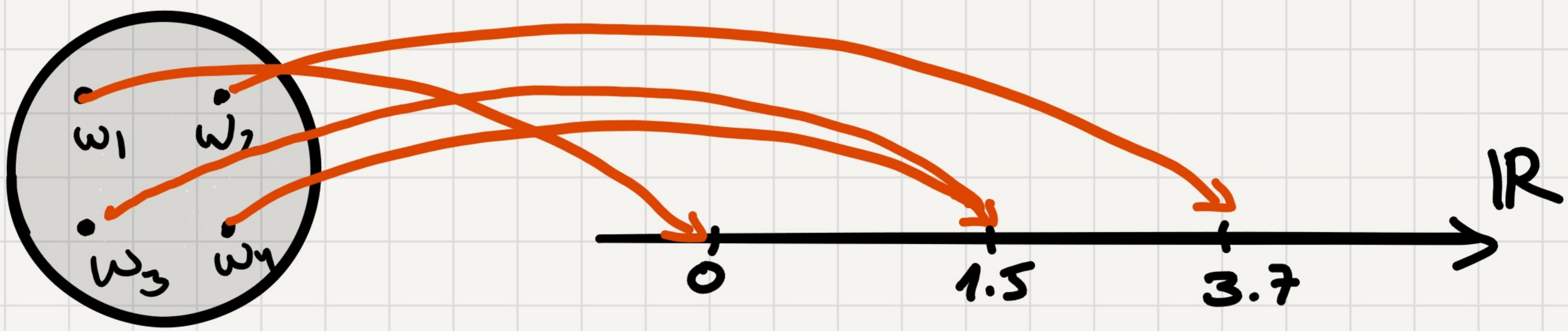
$$X : \Omega \longrightarrow \mathbb{R}$$



Formal Definition

A **random variable**, X, for a random experiment with sample space Ω is a **function** mapping sample points to real numbers.

$$X : \Omega \longrightarrow \mathbb{R}$$

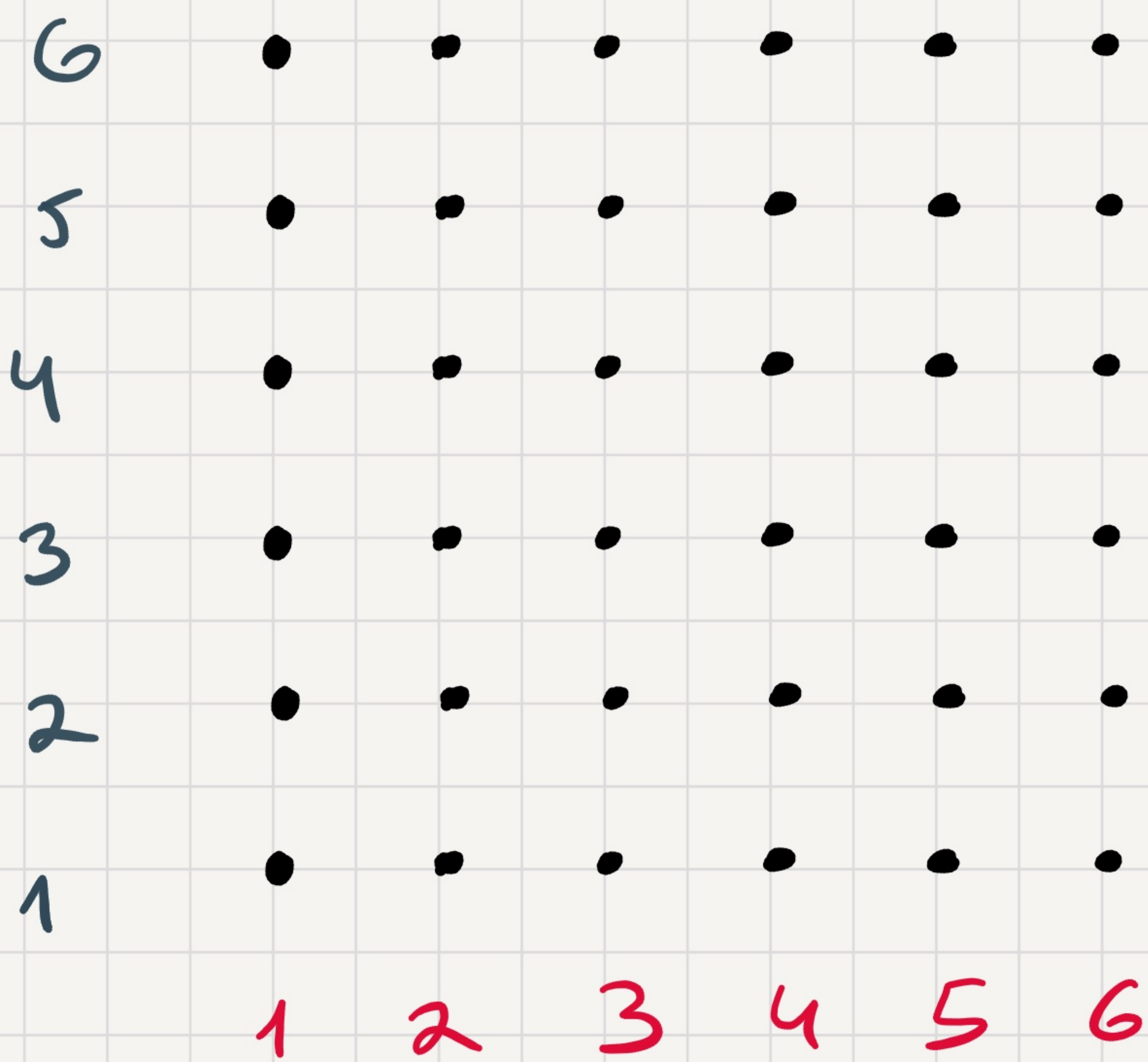


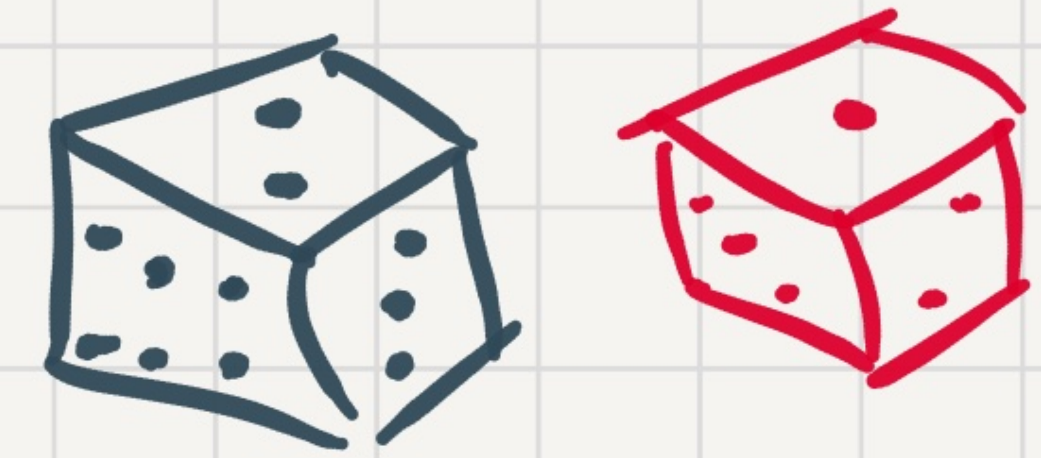
Note: the function is not random.

A **random variable**, X, for a random experiment with sample space Ω is a **function**

$$X : \Omega \longrightarrow \mathbb{R}$$

Example 1: Roll two dice. Q: what's their sum?





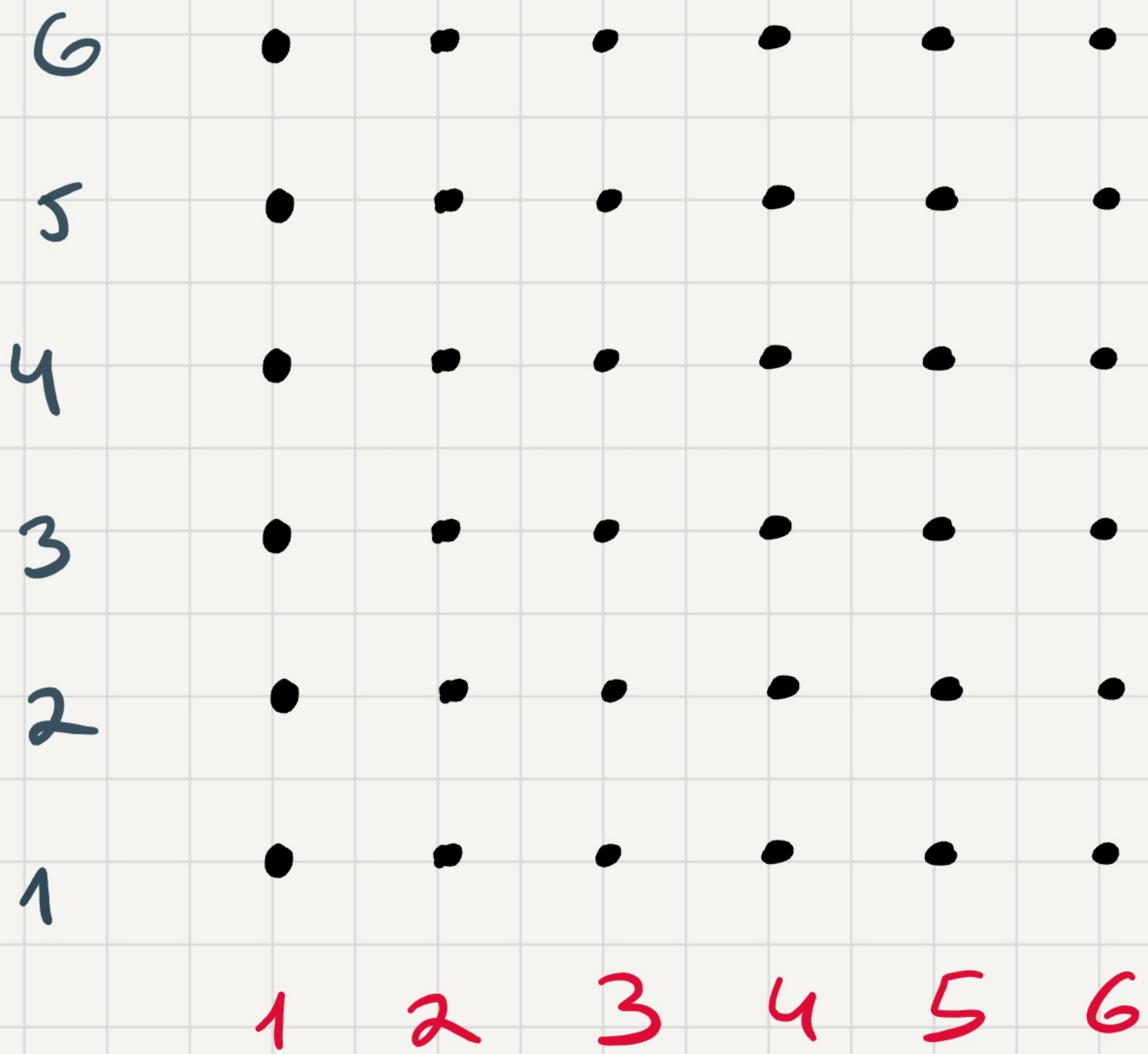
$\Omega = \{(1,1), (1,2), \dots, (6,6)\}$

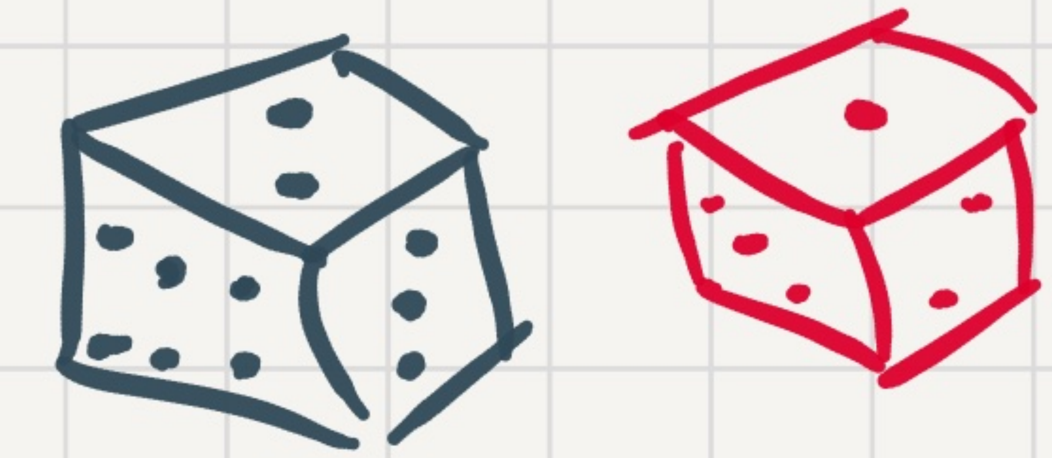
X – the sum

A **random variable**, X, for a random experiment with sample space Ω is a **function**

$$X : \Omega \longrightarrow \mathbb{R}$$

Example 1: Roll two dice. Q: what's their sum?





$\Omega = \{(1,1), (1,2), \ldots, (6,6)\}$

X – the sum

$X(1,1) = 2$

$X(6,1) = 7$

$X(a,b) = a+b$

A random variable, X, for a random experiment with sample space Ω is a function

$$X : \Omega \longrightarrow \mathbb{R}$$

Example 2: toss a 100 coins. X – number of heads

$$\Omega = \{HH\ldots H, HH\ldots HT, \ldots, TT\ldots T\}$$

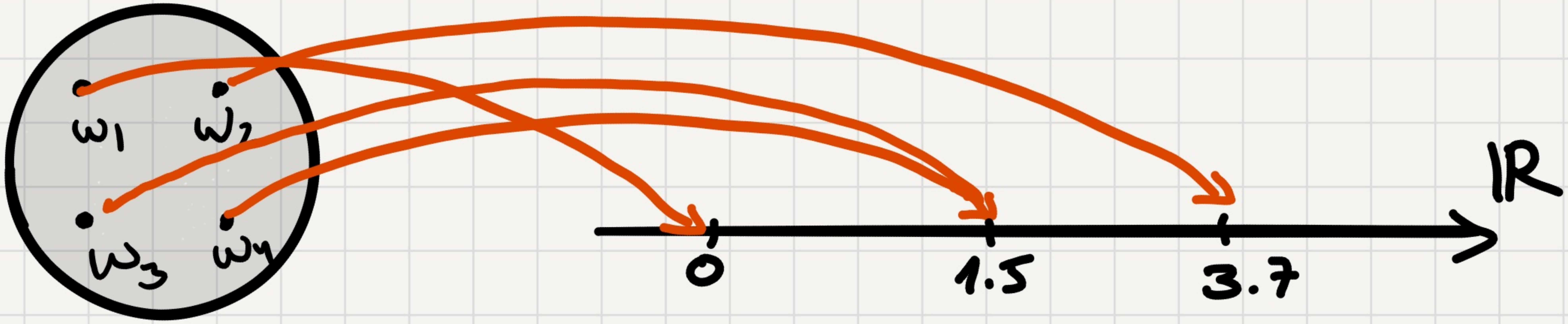
$$X(HHHH\ldots H) =$$

$$X(HH\ldots HT) =$$

$$X(TT\ldots T) =$$

A random variable, X, for a random experiment with sample space Ω is a function

$$X : \Omega \longrightarrow \mathbb{R}$$



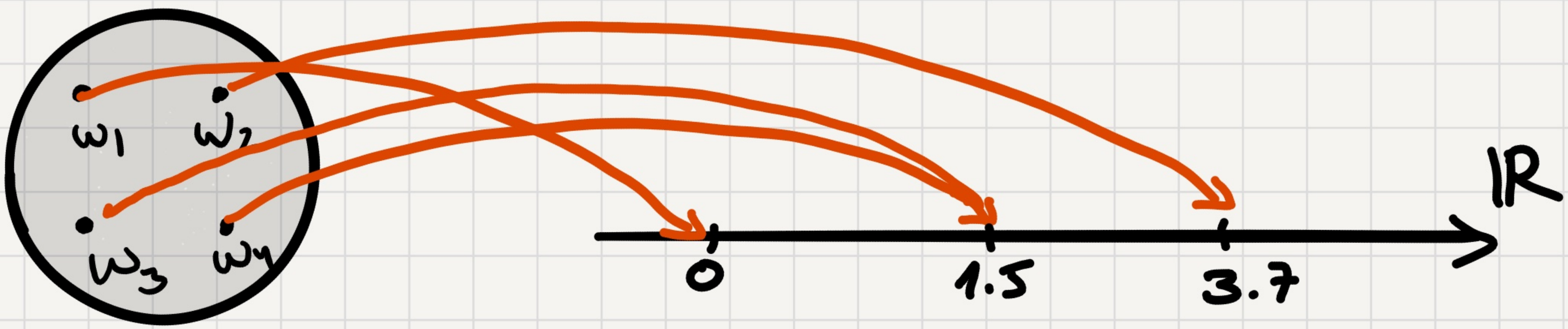
A random variable partitions the sample space to events

$$A_y = \{\omega \in \Omega \mid X(\omega) = y\}$$

e.g., $A_0 = \{\omega_1\}$ $A_{1.5} = \{\omega_3, \omega_4\}$ $A_{3.7} = \{\omega_2\}$

A random variable, X, for a random experiment with sample space Ω is a function

$$X : \Omega \longrightarrow \mathbb{R}$$



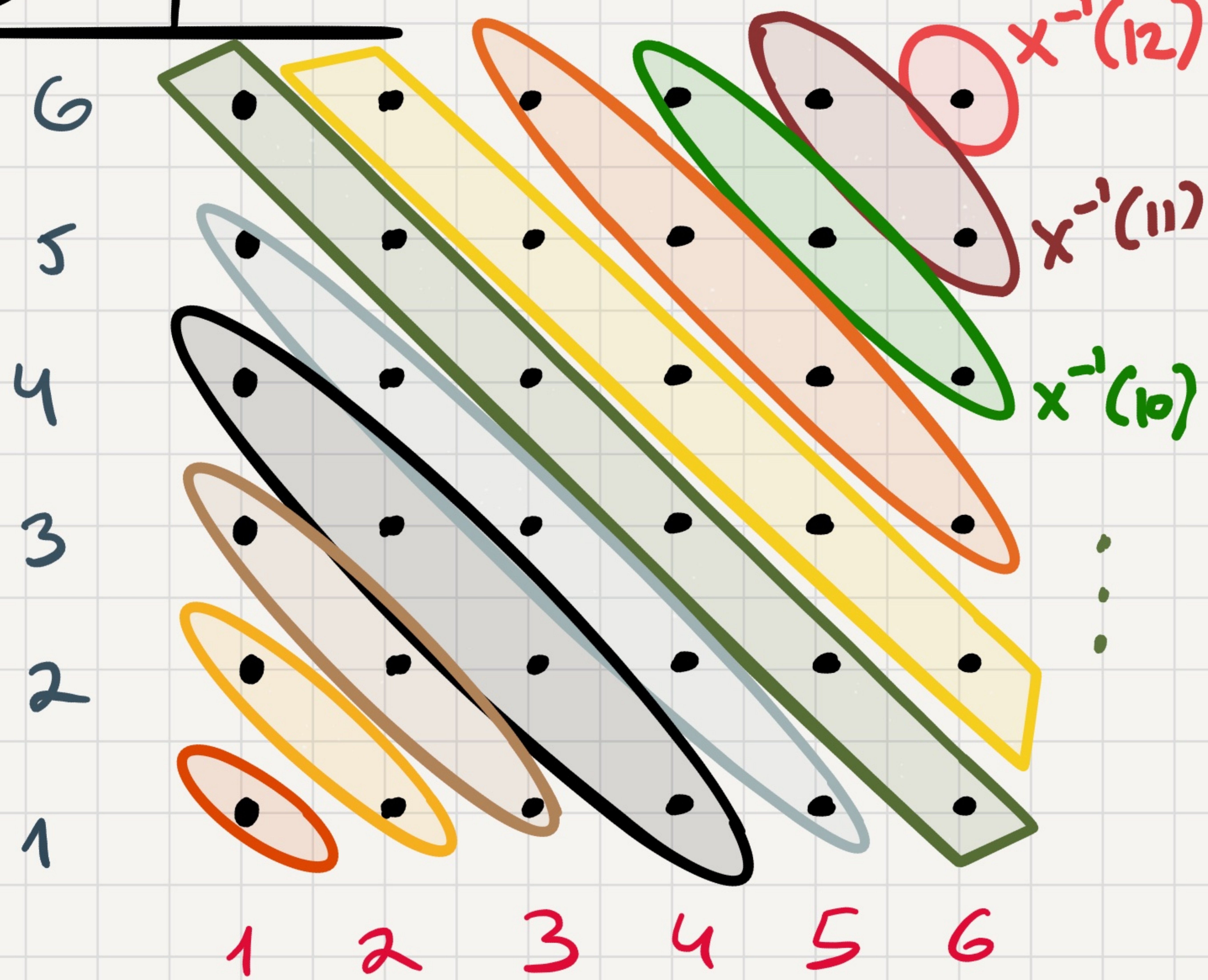
A random variable partitions the sample space to events

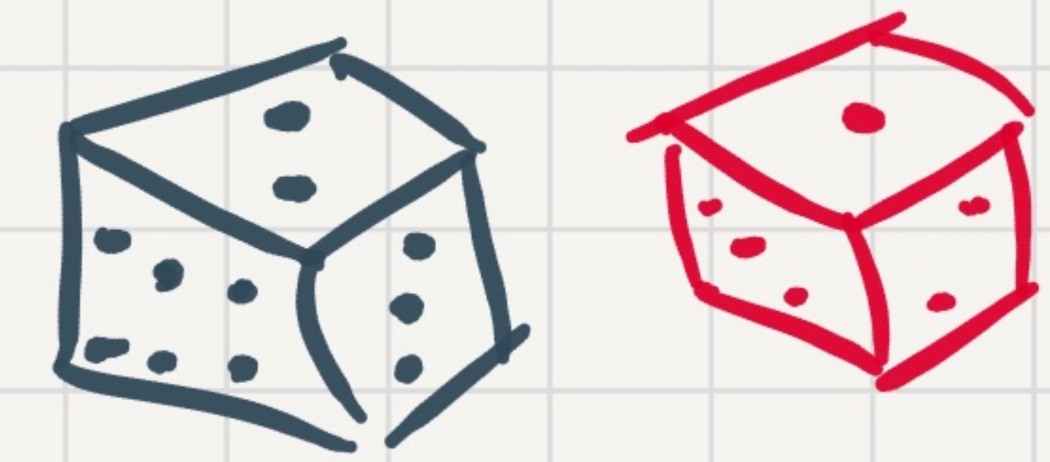
$$A_y = \{\omega \in \Omega \mid X(\omega) = y\} = X^{-1}(y)$$

A <u>random variable</u>, X, for a random experiment with sample space Ω is a <u>function</u>

$$X : \Omega \longrightarrow \mathbb{R}$$

Example 1: Roll two dice. X - their sum.





$Pr[X=10] =$

$Pr[X=8] =$

Distribution

The distribution of a r.v. X is the collection of values

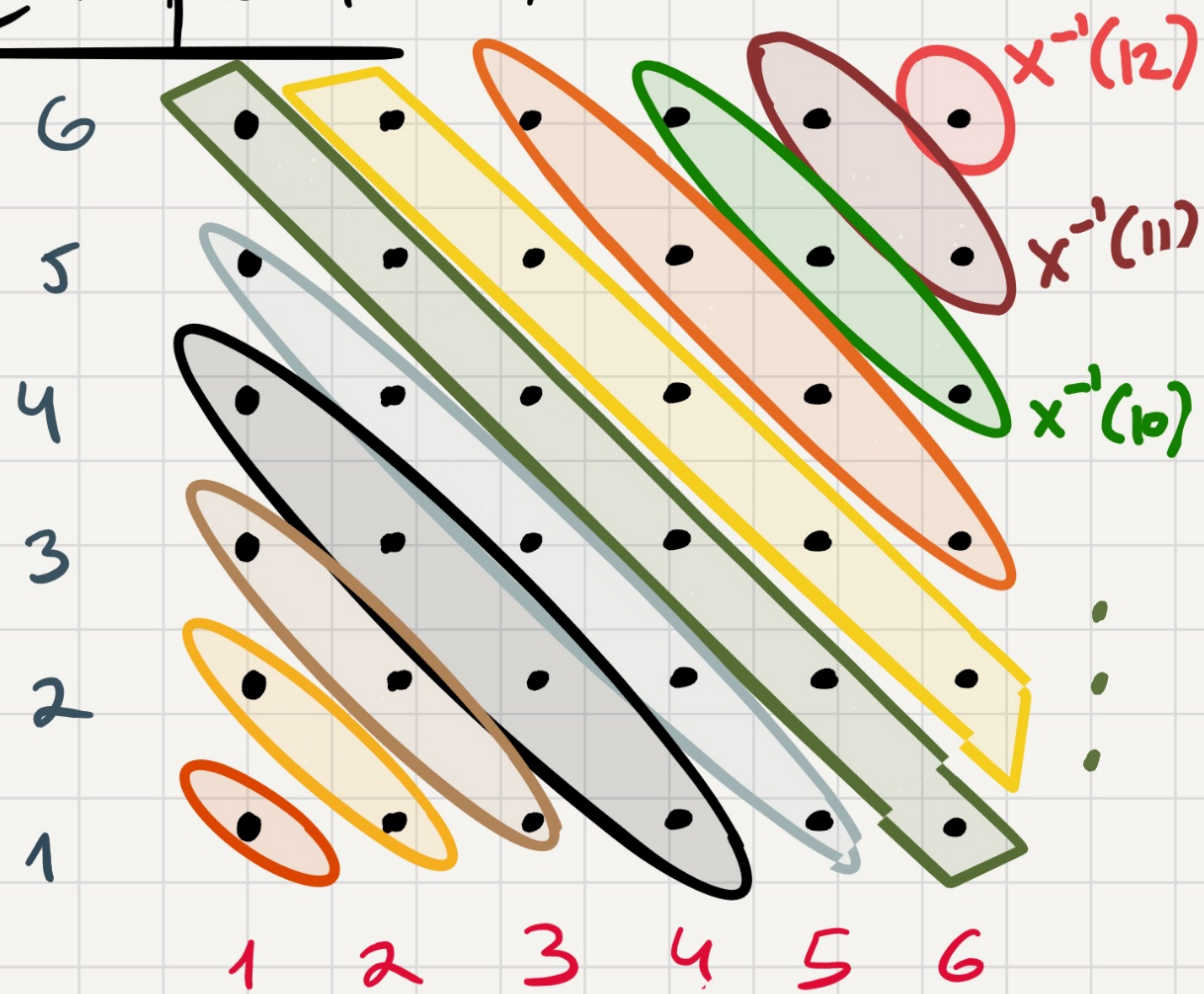
$$\{(a, \Pr[X=a]) : a \in \text{range}(X)\}$$

Distribution

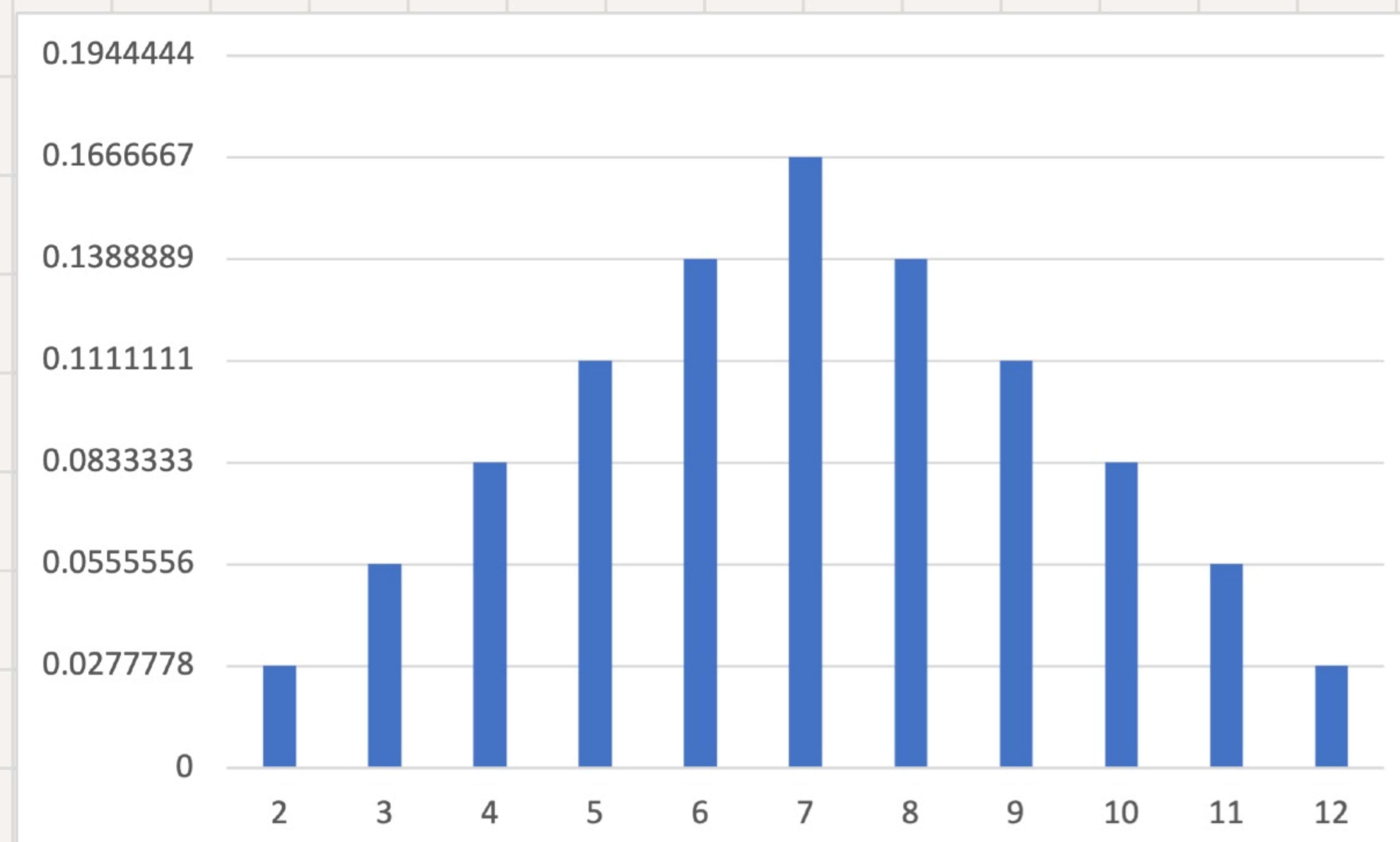
The distribution of a r.v. X is the collection of values

$$\{(a, \Pr[X=a]) : a \in \text{range}(X)\}$$

Example 1: Roll two dice. X - their sum.



The distribution of X:



$\Pr[X=2] = 1/36, \Pr[X=3] = 2/36, \ldots, \Pr[X=12] = \frac{1}{36}$

Example 2: toss a 100 coins. X - number of heads

$$\Omega = \{HH\dots H,\ HH\dots HT,\ \dots,\ TT\dots T\}$$

Distribution of X:

$$\Pr[X = i] =$$

Example 2: toss a 100 coins. X - number of heads

$$\Omega = \{HH\ldots H,\ HH\ldots HT,\ \ldots,\ TT\ldots T\}$$

Distribution of X:

$$\Pr[X=i] = \frac{\#\text{ outcomes with } i \text{ heads}}{\#\text{ outcomes}} = \frac{\binom{100}{i}}{2^{100}}$$

Example 3 : Handing Back Assignments

Random Experiment: hand back assignments to 3 students at random.

X – how many students got back their assignment?

Example 3 : Handing Back Assignments

Random Experiment: hand back assignments to 3 students at random.

X – how many students got back their assignment?

$$\Omega = \{123,\ 132,\ 213,\ 231,\ 312,\ 321\}$$

$$\begin{array}{ccccccc} X & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ & 3 & 1 & 1 & 0 & 0 & 1 \end{array}$$

Example 3: Handing Back Assignments

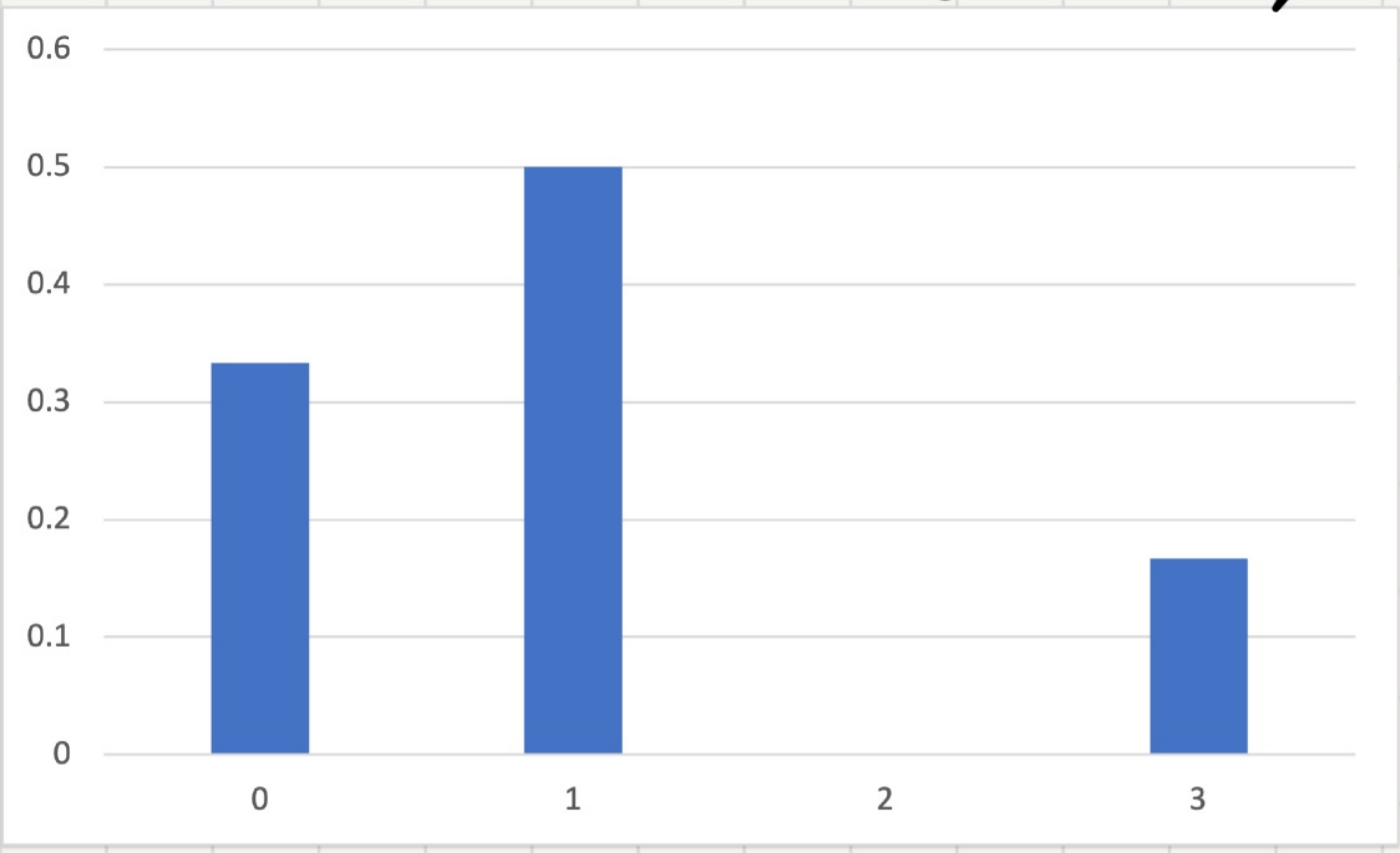
Random Experiment: hand back assignments to 3 students at random.

X – how many students got back their assignment?

$$\Omega = \{\underset{\substack{\downarrow\\3}}{123},\ \underset{\substack{\downarrow\\1}}{132},\ \underset{\substack{\downarrow\\1}}{213},\ \underset{\substack{\downarrow\\0}}{231},\ \underset{\substack{\downarrow\\0}}{312},\ \underset{\substack{\downarrow\\1}}{321}\}$$

X

Distribution of X: $\Pr[X=0] = \frac{2}{6}$, $\Pr[X=1] = \frac{3}{6}$, $\Pr[X=3] = \frac{1}{6}$



Example: Pick a random person in class

$$\Omega = \{ \text{Aaron, Beth, Carol, David, } \ldots \}$$

X - their grade in the midterm.

What's the distribution of X?

Expectation

Expectation Definition

Def'n: The expectation of a r.v. X is defined as

$$\mathbb{E}[X] = \sum_{a \in \text{range}(X)} a \cdot \Pr[X=a]$$

The expected value is also called the mean.

Expectation Definition

<u>Def'n:</u> The expectation of a r.v. X is defined as

$$\mathbb{E}[X] = \sum_{a \in \text{range}(X)} a \cdot \Pr[X=a]$$

<u>Intuition:</u> Suppose we repeat an experiment a large number of times N and let $X_1, \ldots, X_N$ be the results of the r.v.

then $$\frac{X_1 + \ldots + X_N}{N} \approx \mathbb{E}[X]$$

This is because for each value a, the fraction of X_i that equals a approaches $\Pr[X=a]$

Def'n: The expectation of a r.v. X is defined as

$$\mathbb{E}[X] = \sum_{a \in \text{range}(X)} a \cdot \Pr[X = a]$$

Theorem: $$\mathbb{E}[X] = \sum_{\omega \in \Omega} X(\omega) \cdot \Pr[\omega]$$

Def'n: The expectation of a r.v. X is defined as

$$\mathbb{E}[X] = \sum_{a \in \text{range}(X)} a \cdot \Pr[X = a]$$

Theorem: $\mathbb{E}[X] = \sum_{\omega \in \Omega} X(\omega) \cdot \Pr[\omega]$

Proof: $\mathbb{E}[X] = \sum_{a \in \text{range}(X)} a \cdot \Pr[X = a]$

Def'n: The expectation of a r.v. X is defined as

$$\mathbb{E}[X] = \sum_{a \in \text{range}(X)} a \cdot \Pr[X=a]$$

Theorem: $$\mathbb{E}[X] = \sum_{\omega \in \Omega} X(\omega) \cdot \Pr[\omega]$$

Proof:

$$\mathbb{E}[X] = \sum_{a \in \text{range}(X)} a \cdot \Pr[X=a]$$

$$= \sum_{a \in \text{range}(X)} a \cdot \sum_{\omega : X(\omega)=a} \Pr[\omega]$$

Def'n: The expectation of a r.v. X is defined as

$$\mathbb{E}[X] = \sum_{a \in \text{range}(X)} a \cdot \Pr[X=a]$$

Theorem: $$\mathbb{E}[X] = \sum_{\omega \in \Omega} X(\omega) \cdot \Pr[\omega]$$

Proof:
$$\begin{aligned}
\mathbb{E}[X] &= \sum_{a \in \text{range}(X)} a \cdot \Pr[X=a] \\
&= \sum_{a \in \text{range}(X)} a \cdot \sum_{\omega : X(\omega)=a} \Pr[\omega] \\
&= \sum_{a \in \text{range}(X)} \sum_{\omega : X(\omega)=a} X(\omega) \cdot \Pr[\omega]
\end{aligned}$$

Def'n: The expectation of a r.v. X is defined as

$$\mathbb{E}[X] = \sum_{a \in \text{range}(X)} a \cdot \Pr[X = a]$$

Theorem: $$\mathbb{E}[X] = \sum_{\omega \in \Omega} X(\omega) \cdot \Pr[\omega]$$

Proof:

$$\mathbb{E}[X] = \sum_{a \in \text{range}(X)} a \cdot \Pr[X = a]$$

$$= \sum_{a \in \text{range}(X)} a \cdot \sum_{\omega : X(\omega) = a} \Pr[\omega]$$

$$= \sum_{a \in \text{range}(X)} \sum_{\omega : X(\omega) = a} X(\omega) \cdot \Pr[\omega]$$

$$= \sum_{\omega \in \Omega} X(\omega) \cdot \Pr[\omega].$$

$\square$

Example: Pick a random person in class

$$\Omega = \{1, 2, 3, \ldots, 736\}$$

X - their grade in the midterm.

What's the expectation of X?

Example: Pick a random person in class

$$\Omega = \{1, 2, 3, \dots, 736\}$$

X - their grade in the midterm.

What's the expectation of X?

$$\frac{X(1) + X(2) + \dots + X(736)}{736}$$

It's just the average.

More generally expectation is a weighted average.

Example

Toss a fair coin 3 times

$\Omega = \{HHH, HHT, HTH, THH, TTH, THT, HTT, TTT\}$

$X = $ # of H's

what's $\mathbb{E}[X]$?

Example

Toss a fair coin 3 times

$$\Omega = \{\underset{3}{HHH}, \underset{2}{HHT}, \underset{2}{HTH}, \underset{2}{THH}, \underset{1}{TTH}, \underset{1}{THT}, \underset{1}{HTT}, \underset{0}{TTT}\}$$

$X = \#$ of H's

$$\mathbb{E}[X] = \sum_{\omega} X(\omega) \cdot \Pr[\omega] = \frac{3+2+2+2+1+1+1+0}{8}$$

$$\mathbb{E}[X] = \sum_{a} a \cdot \Pr[X=a] = 3 \cdot \frac{1}{8} + 2 \cdot \frac{3}{8} + 1 \cdot \frac{3}{8} + 0 \cdot \frac{1}{8}$$

$$\mathbb{E}[X] = \frac{3}{2}.$$

Famous Distributions

Bernoulli Distribution

X r.v. that takes values in $\{0,1\}$

$$\Pr[X=1] = p$$

$$\Pr[X=0] = 1-p$$

We say that $X \sim \mathrm{Ber}(p)$.

$$\mathbb{E}[X] =$$

Bernoulli Distribution

X r.v. that takes values in $\{0,1\}$

$\Pr[X=1] = p$

$\Pr[X=0] = 1-p$

We say that $X \sim Ber(p)$.

$$\mathbb{E}[X] = p$$

Random Experiment that models it:

Flip a biased coin with heads probability p

$$X(\omega) = \begin{cases} 1, & \omega = H \\ 0, & \omega = T \end{cases}$$

Binomial Distribution

Random Experiment: Flip n biased coins with heads prob. p.

Random Variable: X - number of heads

$$\Omega = \{HHH\cdots H, HHH\cdots HT, \ldots, TTT\cdots T\}$$

Binomial Distribution

Random Experiment: Flip n biased coins with heads prob. p.

Random Variable: X – number of heads

$$\Omega = \{HHH\cdots H, HHH\cdots HT, \ldots, TTT\cdots T\}$$

$n=5$ $\quad \Pr[HHTHT] =$

$\Pr[TTHHH] =$

Binomial Distribution

Random Experiment: Flip n biased coins with heads prob. p.

Random Variable: X – number of heads

$$\Omega = \{HHH\cdots H, HHH\cdots HT, \ldots, TTT\cdots T\}$$

$n=5$ $\quad \Pr[HHTHT] = p \cdot p \cdot (1-p) \cdot p \cdot (1-p) = p^3 \cdot (1-p)^2$

$$\Pr[TTHHH] = (1-p) \cdot (1-p) \cdot p \cdot p \cdot p = p^3 \cdot (1-p)^2$$

$$\Pr[X=i] = (\#\text{ of sequences with } i \text{ heads}) \cdot p^i \cdot (1-p)^{n-i}$$

$$= \binom{n}{i} \cdot p^i (1-p)^{n-i}$$

$$X \sim \mathrm{Bin}(n,p).$$

Binomial Distribution

Random Experiment: Flip n biased coins with heads prob. p.

Random Variable: X – number of heads $\quad X \sim Bin(n,p)$

$$\Omega = \{HHH\cdots H, HHH\cdots HT, \ldots, TTT\cdots T\}$$

$$\Pr[X=i] = (\#\text{ of sequences with } i \text{ heads}) \cdot p^i \cdot (1-p)^{n-i}$$
$$= \binom{n}{i} \cdot p^i (1-p)^{n-i}$$

$$\mathbb{E}[X] = p \cdot n$$

We'll see it next time.

Uniform Distribution

Random Experiment: roll an n-sided die.

$$\Omega = \{1, 2, 3, \ldots, n\}$$

Uniform probability space

$$X(\omega) = \omega.$$

Uniform Distribution

Random Experiment: roll an n-sided die.

$$\Omega = \{1, 2, 3, \ldots, n\}$$

Uniform probability space

$$X(\omega) = \omega.$$

$$\mathbb{E}X = \sum_{i=1}^{n} i \cdot \Pr[X = i]$$

$$= \frac{1 + 2 + \ldots + n}{n} = \frac{1}{n} \times \frac{n \cdot (n+1)}{2} = \frac{n+1}{2}$$

Geometric Distribution

Random Experiment: Flip a coin with $\Pr[H] = p$ until you get H.

Random Variable: X - the number of flips.

What's Ω?

Geometric Distribution

Random Experiment: Flip a coin with $Pr[H]=p$ until you get H.

Random Variable: X - the number of flips.

What's Ω?

Possible outcomes:

$\omega_1 = H$

$\omega_2 = TH$

$\omega_3 = TTH$

$\omega_4 = TTTH$

$\vdots$

$\omega_i = \underbrace{TT\ldots T}_{i-1}H$

$$\Omega = \{\omega_i : i=1,2,\ldots\}$$

Geometric Distribution

Random Experiment: Flip a coin with $Pr[H]=p$ until you get H.

Random Variable: X - the number of flips.

$$\Omega = \{\omega_i : i=1,2,\dots\} \qquad \omega_i = \underbrace{TT\dots T}_{i-1}H$$

$$X(\omega_i)=i$$

$$Pr[X=i] = (1-p)^{i-1}\cdot p \qquad i=1,2,\dots$$

Geometric Distribution

Random Experiment: Flip a coin with $Pr[H]=p$ until you get H.

Random Variable: X - the number of flips.

$$Pr[X=i] = (1-p)^{i-1} \cdot p \qquad i = 1, 2, \dots$$

Let's first see that the prob. sum up to 1.

Geometric Distribution

Random Experiment: Flip a coin with $Pr[H]=p$ until you get H.

Random Variable: X - the number of flips.

$$Pr[X=i] = (1-p)^{i-1} \cdot p \qquad i=1,2,\ldots$$

Let's first see that the prob. sum up to 1.

$$\sum_{i=1}^{\infty} Pr[X=i] = \sum_{i=1}^{\infty} (1-p)^{i-1} \cdot p = p \cdot \sum_{i=1}^{\infty} (1-p)^{i-1} = p \cdot \frac{1}{1-(1-p)} = 1$$

sum of Geom. Series

Geometric Distribution

Random Experiment: Flip a coin with $\Pr[H]=p$ until you get H.

Random Variable: X - the number of flips.

$$\Pr[X=i]=(1-p)^{i-1}\cdot p \qquad i=1,2,\ldots$$

$$\mathbb{E}[X] = p+2(1-p)\cdot p+3(1-p)^2 p+\ldots$$

Geometric Distribution

Random Experiment: Flip a coin with $\Pr[H]=p$ until you get H.

Random Variable: X - the number of flips.

$$\Pr[X=i] = (1-p)^{i-1} \cdot p \qquad i=1,2,\ldots$$

$$\mathbb{E}[X] = p + 2(1-p)\cdot p + 3(1-p)^2 p + \ldots$$

$$(1-p)\cdot \mathbb{E}[X] = (1-p)\cdot p + 2(1-p)^2 \cdot p + \ldots$$

Geometric Distribution

Random Experiment: Flip a coin with $\Pr[H]=p$ until you get H.

Random Variable: X - the number of flips.

$$\Pr[X=i] = (1-p)^{i-1}\cdot p \qquad i=1,2,\dots$$

$$-\begin{cases} \mathbb{E}[X] = p + 2(1-p)\cdot p + 3(1-p)^2 p + \dots \\ (1-p)\cdot \mathbb{E}[X] = (1-p)\cdot p + 2(1-p)^2\cdot p + \dots \end{cases}$$

$$p\cdot\mathbb{E}[X] = p + (1-p)\cdot p + (1-p)^2\cdot p + \dots = 1$$

So $\mathbb{E}[X] = 1/p$.

Poisson Distribution

Q: How many customer arrive to McDonalds in 1 hour?

Suppose you know: the **average** is λ.

Poisson Distribution

Q: How many customer arrive to McDonalds in 1 hour?

Suppose you know: the **average** is λ.

Assumption: Arrivals in disjoint time intervals are independent.

Poisson Distribution

Q: How many customer arrive to McDonalds in 1 hour?

Suppose you know: the average is λ.

Assumption: Arrivals in disjoint time intervals are independent.

Idea: Cut 1 hour to equal intervals of length $\frac{1}{n}$ for n extremely large.

Average arrivals per interval $\frac{\lambda}{n}$.

Poisson Distribution

Q: How many customer arrive to McDonalds in 1 hour?

Suppose you know: the **average** is λ.

Assumption: Arrivals in disjoint time intervals are independent.

Idea: Cut 1 hour to equal intervals of length $\frac{1}{n}$ for n extremely large.

Average arrivals per interval $\frac{\lambda}{n}$.

Assumption 2: no two arrivals in the same interval.

Poisson Distribution

Q: How many customer arrive to McDonalds in 1 hour?

Suppose you know: the average is λ.

Assumption: Arrivals in disjoint time intervals are independent.

Idea: Cut 1 hour to equal intervals of length $\frac{1}{n}$ for n extremely large.

Average arrivals per interval $\frac{\lambda}{n}$.

Assumption 2: no two arrivals in the same interval.

Model: Binomial distribution with params $(n, \frac{\lambda}{n})$.

Next Time

Fix λ, i. Let $X \sim Bin(n, \frac{\lambda}{n})$. Then,

$$\Pr_{X \sim Bin(n, \frac{\lambda}{n})}[X = i] \xrightarrow[n \to \infty]{} e^{-\lambda} \cdot \frac{\lambda^i}{i!}$$

these define

the Poisson distribution

Summary

- A r.v. is a function $X : \Omega \to \mathbb{R}$.
- A r.v. induces a partition on the sample space to events $X^{-1}(y) = \{\omega \in \Omega : X(\omega) = y\}$
- A distribution of a r.v. is the collection of values
$$\{(a, \Pr[X=a]) : a \in \text{range}(X)\}$$
- The expectation of a r.v. X is defined as
$$\mathbb{E}[X] = \sum_{a \in \text{range}(X)} a \cdot \Pr[X=a]$$
- $\text{Ber}(p)$ – one trial, success prob. p.
- $\text{Bin}(n,p)$ – n trials, " " p.
- Uniform Dist., Geometric Dist., Poisson Dist.