Lecture 19

Random Variables II

Plan for Today:

- Poisson Distribution (continued)
- Joint Distributions
- Linearity of Expectation
- Independence

Lecture 18 Summary

- A r.v. is a function $X : \Omega \to \mathbb{R}$.
- A r.v. induces a partition on the sample space to events $X^{-1}(a) = \{\omega \in \Omega : X(\omega) = a\}$
- A distribution of a r.v. is the collection of values $\{(a, \Pr[X=a]) : a \in \text{range}(X)\}$
- The expectation of a r.v. X is defined as
$$\mathbb{E}[X] = \sum_{a \in \text{range}(X)} a \cdot \Pr[X=a]$$
- $\text{Ber}(p)$ – one trial, success probability p.
- $\text{Bin}(n,p)$ – n trials, success probability p
- $\text{Geom}(p)$ – number of trials upto first success, suc. prob. p.
- uniform distribution, Poisson distribution

Poisson Distribution

Q: How many customers arrive to McDonalds in 1 hour?

Suppose you know: the average is λ.

Assumption: Arrivals in disjoint time intervals are independent.

Idea: Cut 1 hour to equal intervals of length $\frac{1}{n}$ for n extremely large.

Average arrivals per interval $\frac{\lambda}{n}$.

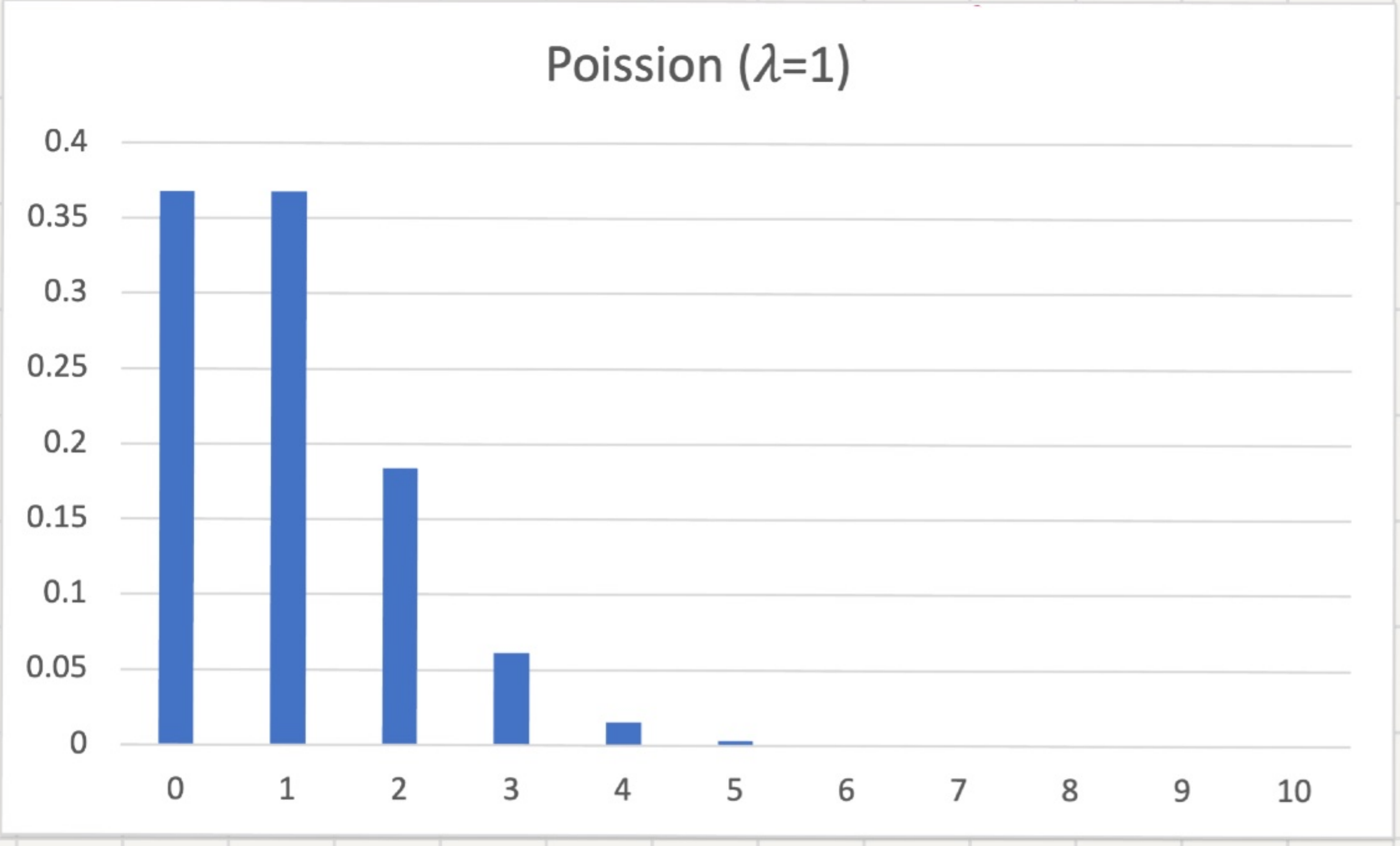
Assumption 2: no two arrivals in the same interval.

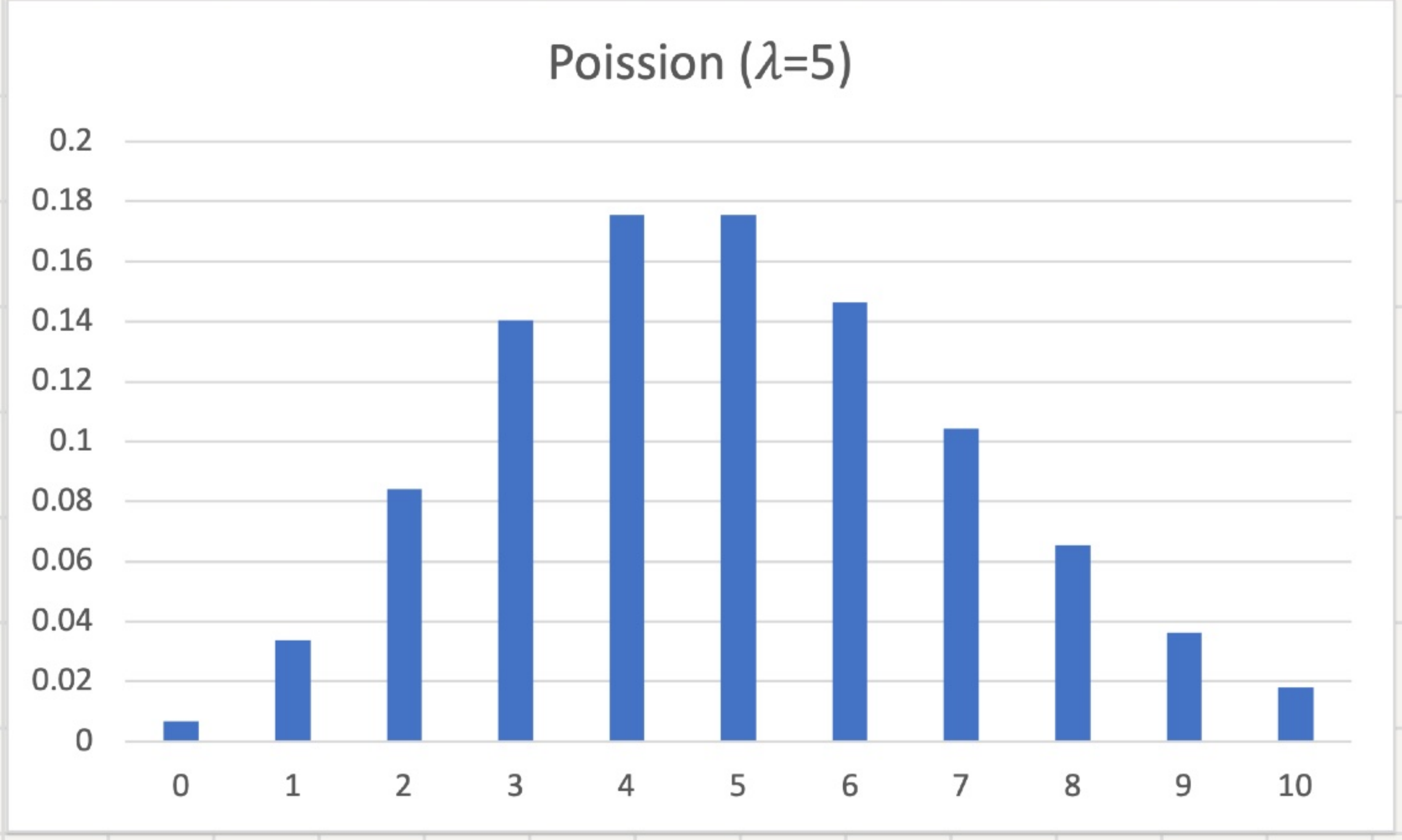
Model: Binomial distribution with params $(n, \frac{\lambda}{n})$.

Theorem: Fix λ, i. Let $X \sim \mathrm{Bin}(n, \frac{\lambda}{n})$. Then,

$$\Pr[X = i] \xrightarrow[n \to \infty]{} e^{-\lambda} \cdot \frac{\lambda^i}{i!}$$

these define the Poisson distribution





Theorem: Fix λ, i. Let $X \sim Bin(n, \frac{\lambda}{n})$. Then,

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Proof:

Theorem: Fix λ, i. Let $X \sim \mathrm{Bin}(n, \frac{\lambda}{n})$. Then,

$$\Pr[X = i] \xrightarrow[n \to \infty]{} e^{-\lambda} \cdot \frac{\lambda^i}{i!}$$

Proof:

$$\Pr[X = i] = \binom{n}{i} \cdot p^i \cdot (1-p)^{n-i}$$

$$= \frac{n \cdot (n-1) \cdots (n-i+1)}{i!} \cdot \frac{\lambda^i}{n^i} \cdot \left(1 - \frac{\lambda}{n}\right)^{n-i}$$

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$$= \frac{n\cdot(n-1)\cdots(n-i+1)}{i!} \cdot \frac{\lambda^i}{n^i} \cdot \left(1-\frac{\lambda}{n}\right)^{n-i}$$

$$= \frac{n\cdot(n-1)\cdots(n-i+1)}{n\cdot n \cdots n} \cdot \frac{\lambda^i}{i!} \cdot \frac{\left(1-\frac{\lambda}{n}\right)^n}{\left(1-\frac{\lambda}{n}\right)^i}$$

$$\xrightarrow[n\to\infty]{}$$

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$$= \frac{n \cdot (n-1) \cdots (n-i+1)}{n \cdot n \cdots n} \cdot \frac{\lambda^i}{i!} \cdot \frac{\left(1 - \frac{\lambda}{n}\right)^n}{\left(1 - \frac{\lambda}{n}\right)^i}$$

$$\xrightarrow[n \to \infty]{} 1 \cdot \frac{\lambda^i}{i!} \cdot \frac{e^{-\lambda}}{1}$$

$\blacksquare$

In the last line we used $\lim_{n \to \infty} \left(1 - \frac{\lambda}{n}\right)^n = e^{-\lambda}$.

Expectation

If X is a r.v. then

$$\mathbb{E}[X] = \sum_{a \in \text{range}(X)} a \cdot \Pr[X = a]$$

Theorem: If $X : \Omega \to \mathbb{R}$ is a r.v. then

$$\mathbb{E}[X] = \sum_{\omega \in \Omega} X(\omega) \cdot \Pr[\omega]$$

Example: Toss three coins. $X = \#$ of heads $- \#$ of tails

$\Omega = \{HHH, HHT, HTH, THH, TTH, THT, HTT, TTT\}$

Expectation

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Example: Toss three coins. $X = \#$ of heads $- \#$ of tails

$\Omega = \{HHH, HHT, HTH, THH, TTH, THT, HTT, TTT\}$

$X = \quad 3, \quad 1, \quad 1, \quad 1, \quad -1, \quad -1, \quad -1, \quad -3$

$$\mathbb{E}[X] = 3 \cdot \frac{1}{8} + 1 \cdot \frac{3}{8} + (-1) \cdot \frac{3}{8} + (-3) \cdot \frac{1}{8} = 0$$

$$\mathbb{E}[X] = \frac{3 + 1 + 1 + 1 + (-1) + (-1) + (-1) + (-3)}{8} = 0$$

Expectation of a Poisson R.V.

If X is a r.v. then

$$\mathbb{E}[X] = \sum_{a \in \text{range}(X)} a \cdot \Pr[X=a]$$

Theorem: If $X : \Omega \to \mathbb{R}$ is a r.v. then

$$\mathbb{E}[X] = \sum_{\omega \in \Omega} X(\omega) \cdot \Pr[\omega]$$

X is a Poisson r.v. with parameter λ. $(X \sim \text{Pois}(\lambda))$

For any $i = 0, 1, \ldots$ $\quad \Pr[X=i] = e^{-\lambda} \cdot \frac{\lambda^i}{i!}$

$$\mathbb{E}[X] = ?$$

Expectation of a Poisson R.V.

If X is a r.v. then

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For any $i = 0, 1, \ldots$ $\quad \Pr[X=i] = e^{-\lambda} \cdot \frac{\lambda^i}{i!}$

$$\mathbb{E}[X] = \sum_{i=0}^{\infty} i \cdot \Pr[X=i] = \sum_{i=1}^{\infty} i \cdot e^{-\lambda} \frac{\lambda^i}{i!} = \sum_{i=1}^{\infty} e^{-\lambda} \cdot \frac{\lambda^i}{(i-1)!}$$

$$= \lambda \cdot \sum_{i=1}^{\infty} e^{-\lambda} \cdot \frac{\lambda^{i-1}}{(i-1)!} = \lambda \sum_{i=0}^{\infty} e^{-\lambda} \cdot \frac{\lambda^i}{i!} = \lambda.$$

Multiple Random Variables

Experiment: toss 2 coins

$$\Omega = \{HH, HT, TH, TT\}$$

$$X(\omega) = \begin{cases} 1, & \text{if coin \#1 is heads} \\ 0, & \text{o.w.} \end{cases}$$

$$Y(\omega) = \begin{cases} 1, & \text{if coin \#2 is heads} \\ 0, & \text{o.w.} \end{cases}$$

X and Y are two different r.v.s on the same sample space.

Multiple Random Variables

Joint Distribution: If X and Y are two r.v.s over the same probability space then their joint distribution is defined as

$$\{(a, b, \Pr[X=a, Y=b]) : a \in \text{range}(X),\ b \in \text{range}(Y)\}$$

Multiple Random Variables

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Example Flip two coins

$X = 1$ iff first coin is heads
$Y = 1$ iff second coin is heads

X \ Y	0	1
0	1/4	1/4
1	1/4	1/4

Multiple Random Variables

Joint Distribution: If X and Y are two r.v.s over the same probability space then their joint distribution is defined as

$$\{(a, b, \Pr[X=a, Y=b]) : a \in \text{range}(X), b \in \text{range}(Y)\}$$

Example Flip two biased coins with heads prob. p | $X=1$ iff first coin is H, $Y=1$ iff second coin is H

$X \setminus Y$	0	1
0	$(1-p)^2$	$(1-p)\cdot p$
1	$p\cdot(1-p)$	p^2

Multiple Random Variables

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$$\{(a, b, \Pr[X=a, Y=b]) : a \in \text{range}(X), b \in \text{range}(Y)\}$$

Marginal Distributions

Marginal for X: $\Pr[X=a] = \sum_{b \in \text{range}(Y)} \Pr[X=a, Y=b]$

Marginal for Y: $\Pr[Y=b] = \sum_{a \in \text{range}(X)} \Pr[X=a, Y=b]$

Multiple Random Variables

Joint Distribution:

$$\{(a, b, Pr[X=a, Y=b]) : a \in range(X), b \in range(Y)\}$$

Marginal for X: $Pr[X=a] = \sum_{b \in range(Y)} Pr[X=a, Y=b]$

Marginal for Y: $Pr[Y=b] = \sum_{a \in range(X)} Pr[X=a, Y=b]$

Example:

X \ Y	1	2	3
1	0	0.1	0.2
2	0.3	0	0
3	0.1	0.2	0.1

$Pr[X=1] =$

$Pr[Y=3] =$

Multiple Random Variables

Joint Distribution:

$$\{(a, b, \Pr[X=a, Y=b]) : a \in \text{range}(X), b \in \text{range}(Y)\}$$

Marginal for X: $\Pr[X=a] = \sum_{b \in \text{range}(Y)} \Pr[X=a, Y=b]$

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Example:

X \ Y	1	2	3
1	0	0.1	0.2
2	0.3	0	0
3	0.1	0.2	0.1

$\Pr[X=1] =$

$\Pr[Y=3] =$

$\Pr[X=1 \mid Y=3] =$

Recap

Independence

Definition: We say that events A and B are **independent** if

$$\Pr[A \cap B] = \Pr[A] \cdot \Pr[B].$$

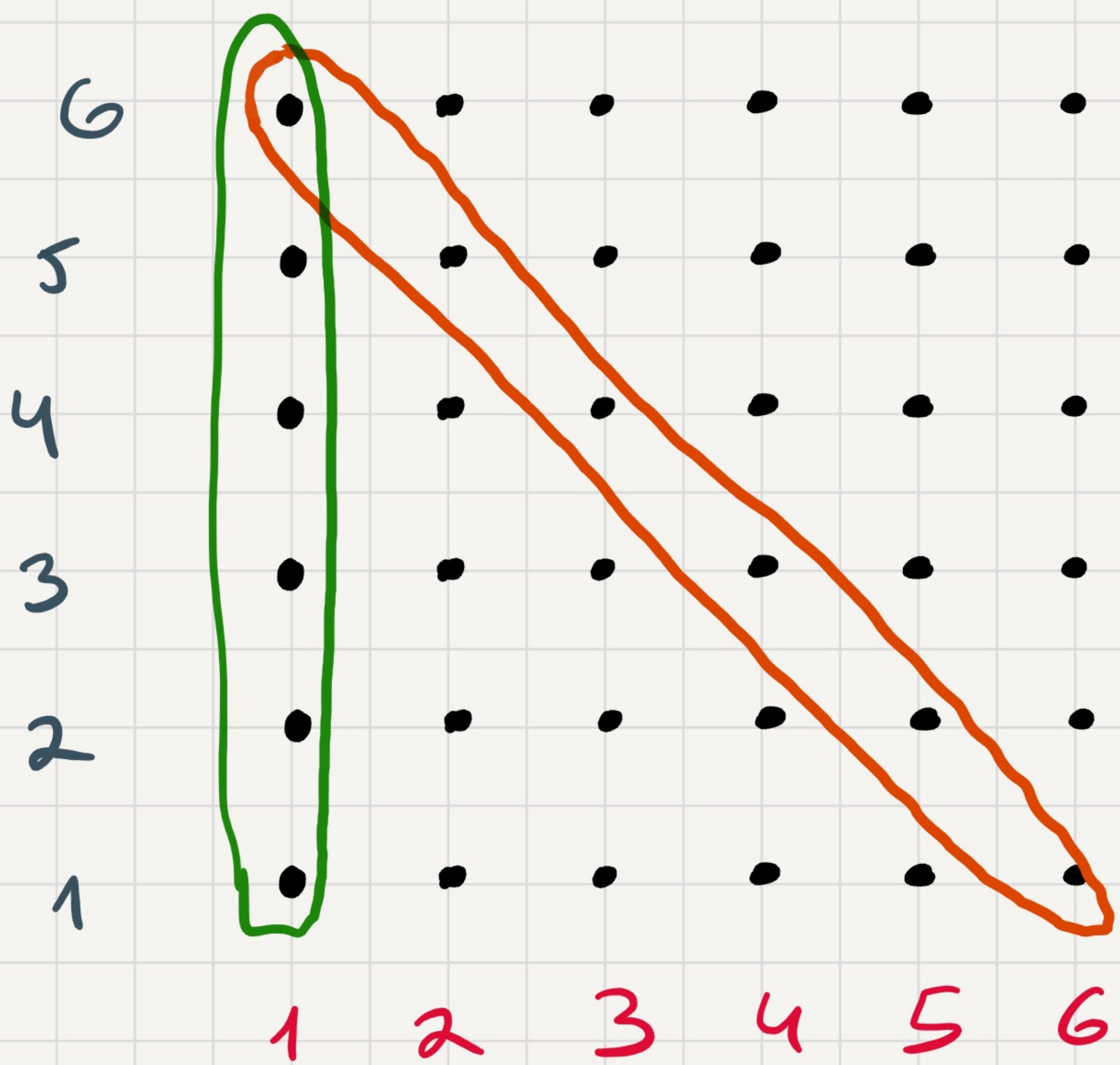
or equivalently $\quad \Pr[B \mid A] = \Pr[B]$

" $\quad \Pr[A \mid B] = \Pr[A].$

Recap

Rolling two dice

A = "sum is 7" B = " red die is 1"



Pr. of each outcome $= 1/36$.

$$\Pr[A \cap B] = 1/36 = \Pr[A] \cdot \Pr[B]$$

independent!

Recap

Definition Pairwise Independence

We say that events $A_1, \dots, A_n$ are pairwise independent if

$\forall i \neq j$ A_i and A_j are independent.

Definition Mutual Independence

We say that events A_1, A_2, A_3 are mutually independent if they are pairwise indep. and $\Pr[A_1 \cap A_2 \cap A_3]$
$= \Pr[A_1] \cdot \Pr[A_2] \cdot \Pr[A_3]$

Recap

Definition Pairwise Independence

We say that events $A_1, \ldots, A_n$ are pairwise independent if

$\forall\, i \neq j$ A_i and A_j are independent.

Definition Mutual Independence

We say that events $A_1, \ldots, A_n$ are mutually independent if for each non-empty subset $I \subseteq \{1, \ldots, n\}$ $\Pr\left[\bigcap_{i \in I} A_i\right] = \prod_{i \in I} \Pr[A_i]$

Independent Random Variables

Def'n: We say that two r.v.s X and Y are independent if

$$\Pr[X=a, Y=b] = \Pr[X=a] \cdot \Pr[Y=b]$$

for all $a \in \text{range}(X)$
$b \in \text{range}(Y)$.

Independent Random Variables

Def'n: We say that two r.v.s X and Y are independent if

$$\Pr[X=a, Y=b] = \Pr[X=a] \cdot \Pr[Y=b]$$

for all $a \in \text{range}(X)$
$b \in \text{range}(Y)$.

Fact: If X & Y are indep.

$$\Pr[X=a \mid Y=b] = \Pr[X=a]$$

Independence - Examples

Example 1: Roll two dice. X - result of first die
Y - " " second "

Example 2: Roll two dice. X - result of first die
Y - sum of two dice.

Example 3: Flip a fair coin 10 times
X - number of heads in first 5 flips
Y - " " " " last 5 flips.

Independence - Examples

Example 1: Roll two dice. X - result of first die
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Are X & Y independent?

Independence – Examples

Example 1: Roll two dice. X – result of first die
Y – " " second "

Are X & Y independent? Yes.

for $a, b \in \{1, \ldots, 6\}$: $\Pr[X=a, Y=b] = \frac{1}{36} = \Pr[X=a] \cdot \Pr[Y=b]$

Example 2: Roll two dice. X - result of first die
Y - sum of two dice.

Are X & Y Independent?

Example 3: Flip a fair coin 10 times

X - number of heads in first 5 flips

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Are X & Y independent?

Example 3: Flip a fair coin 10 times

X - number of heads in first 5 flips

Y - " " " " last 5 flips.

Are X & Y independent? Yes

for $a, b \in \{0, 1, \dots, 5\}$

$$\Pr[X=a, Y=b] = \binom{5}{a}\binom{5}{b} \cdot \frac{1}{2^{10}} = \binom{5}{a}\frac{1}{2^5} \cdot \binom{5}{b} \cdot \frac{1}{2^5}$$

$$= \Pr[X=a] \cdot \Pr[Y=b]$$

Linearity of Expectation

Theorem: If X and Y are r.v.s (over the same prob. space)

then $$\mathbb{E}[X+Y] = \mathbb{E}[X] + \mathbb{E}[Y]$$

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Note: No need for X & Y to be independent!

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then $\mathbb{E}[X+Y] = \mathbb{E}[X] + \mathbb{E}[Y]$

Note: No need for X & Y to be independent!

Proof:

$$\begin{aligned}
\mathbb{E}[X+Y] &= \sum_{\omega \in \Omega} (X+Y)(\omega) \cdot \Pr[\omega] \\
&= \sum_{\omega \in \Omega} \big(X(\omega) + Y(\omega)\big) \cdot \Pr[\omega] \\
&= \Big(\sum_{\omega \in \Omega} X(\omega) \cdot \Pr[\omega]\Big) + \Big(\sum_{\omega \in \Omega} Y(\omega) \cdot \Pr[\omega]\Big) \\
&= \mathbb{E}[X] + \mathbb{E}[Y].
\end{aligned}$$

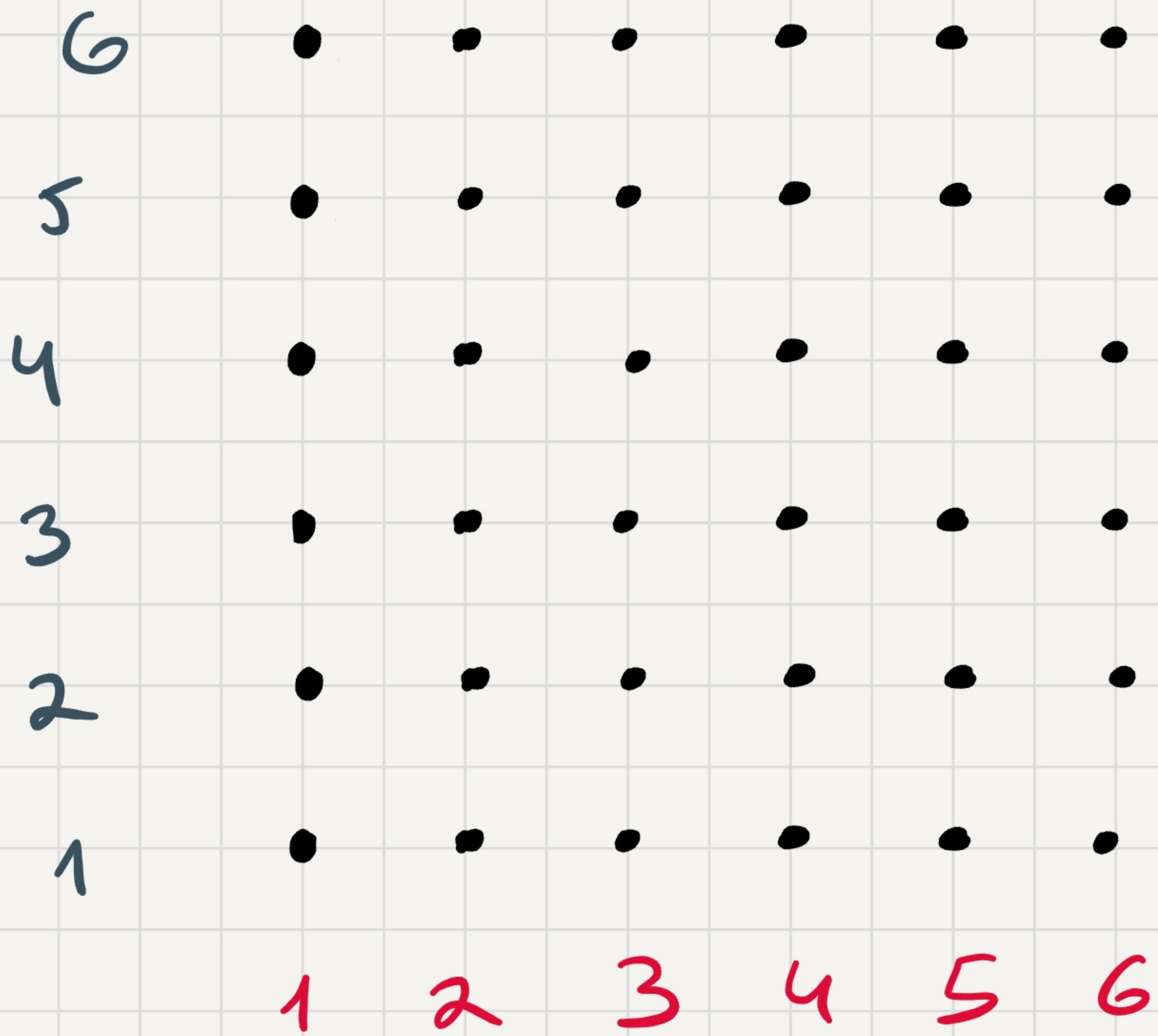
Theorem: If $X_1, \dots, X_n$ are r.v.s $a_1, \dots, a_n \in \mathbb{R}$ then

$$\mathbb{E}[a_1 X_1 + \dots + a_n X_n] = a_1 \mathbb{E}[X_1] + \dots + a_n \mathbb{E}[X_n]$$

Proof: Induction on n.

Rolling two dice

X - the sum of the two dice



Pr. of each outcome $= 1/36$.

Dist. of X

$$\Pr[X=a] = \begin{cases} 1/36, & a=2 \\ 2/36, & a=3 \\ \vdots & \\ 1/36, & a=12 \end{cases}$$

$$\mathbb{E}[X] = 2 \cdot \frac{1}{36} + 3 \cdot \frac{2}{36} + 4 \cdot \frac{3}{36} + \cdots + 12 \cdot \frac{1}{36}$$

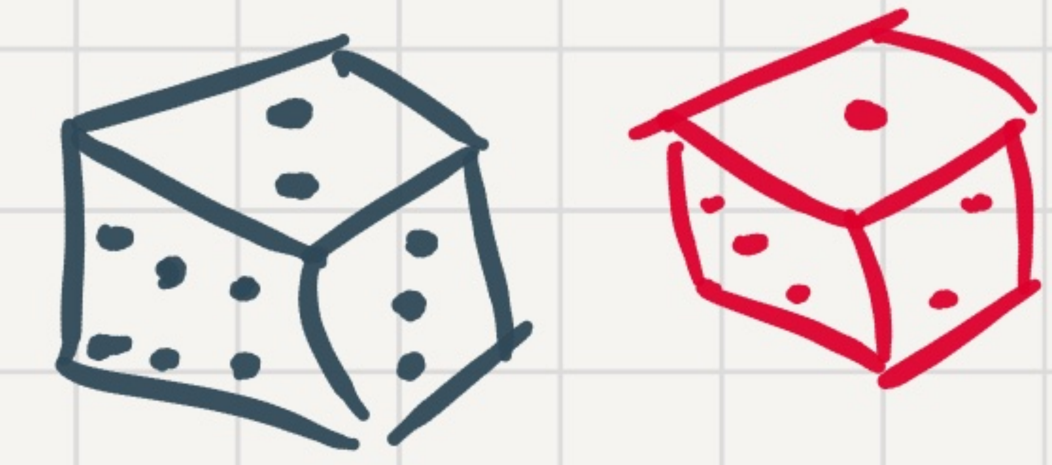
$$= 7.$$

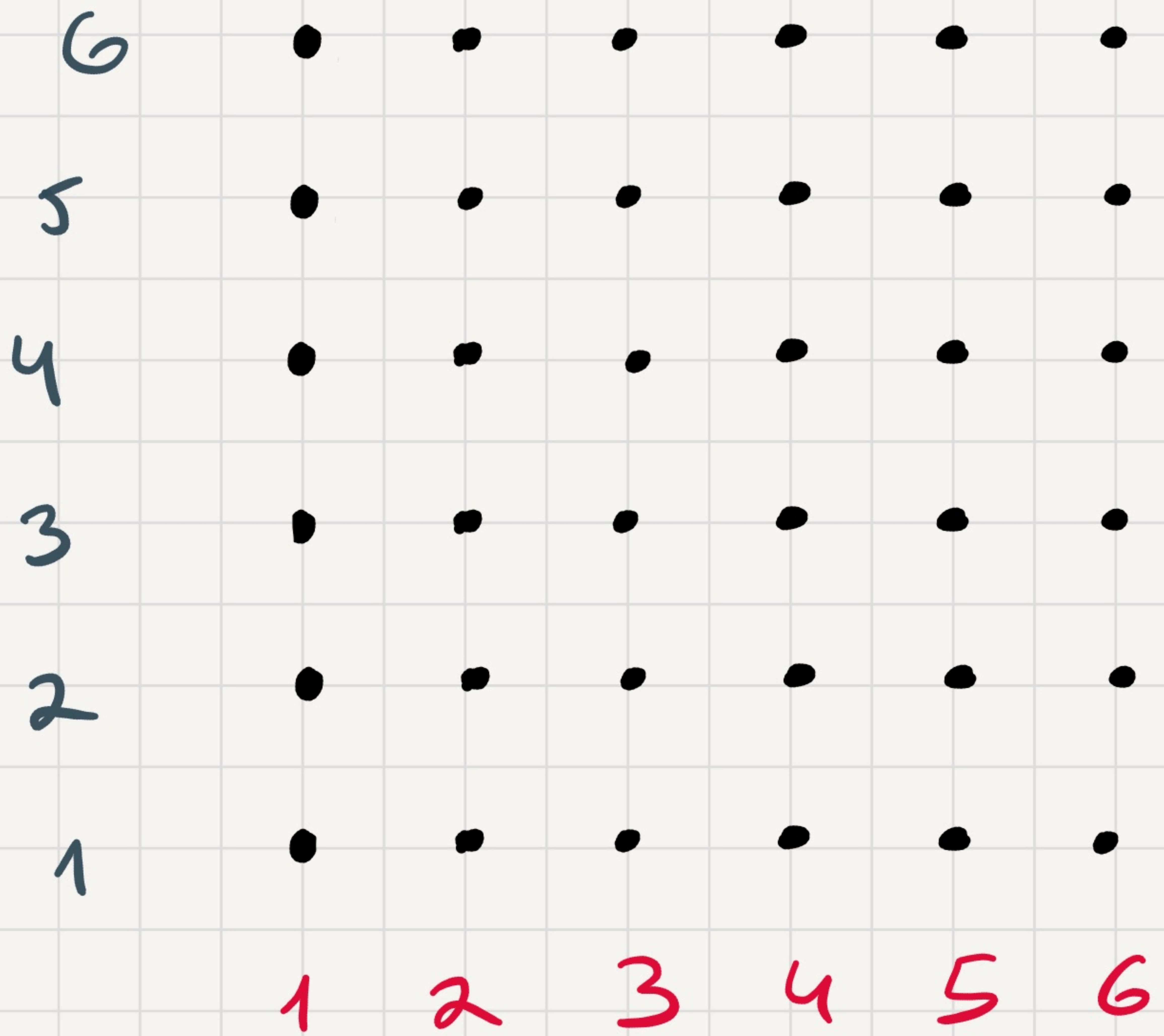
Rolling two dice

X_1 – result of first die

X_2 – " " second die

$X = X_1 + X_2$





Pr. of each outcome $= 1/36$.

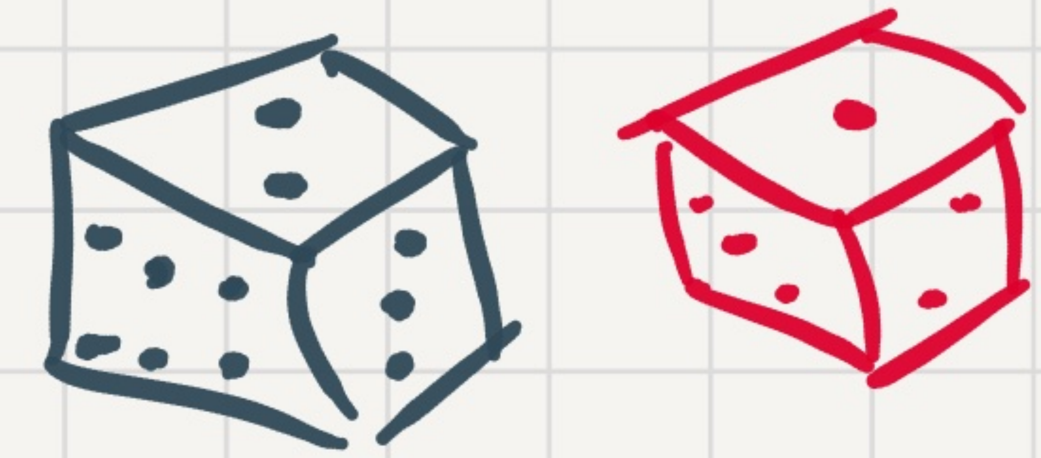
$\mathbb{E}[X] =$

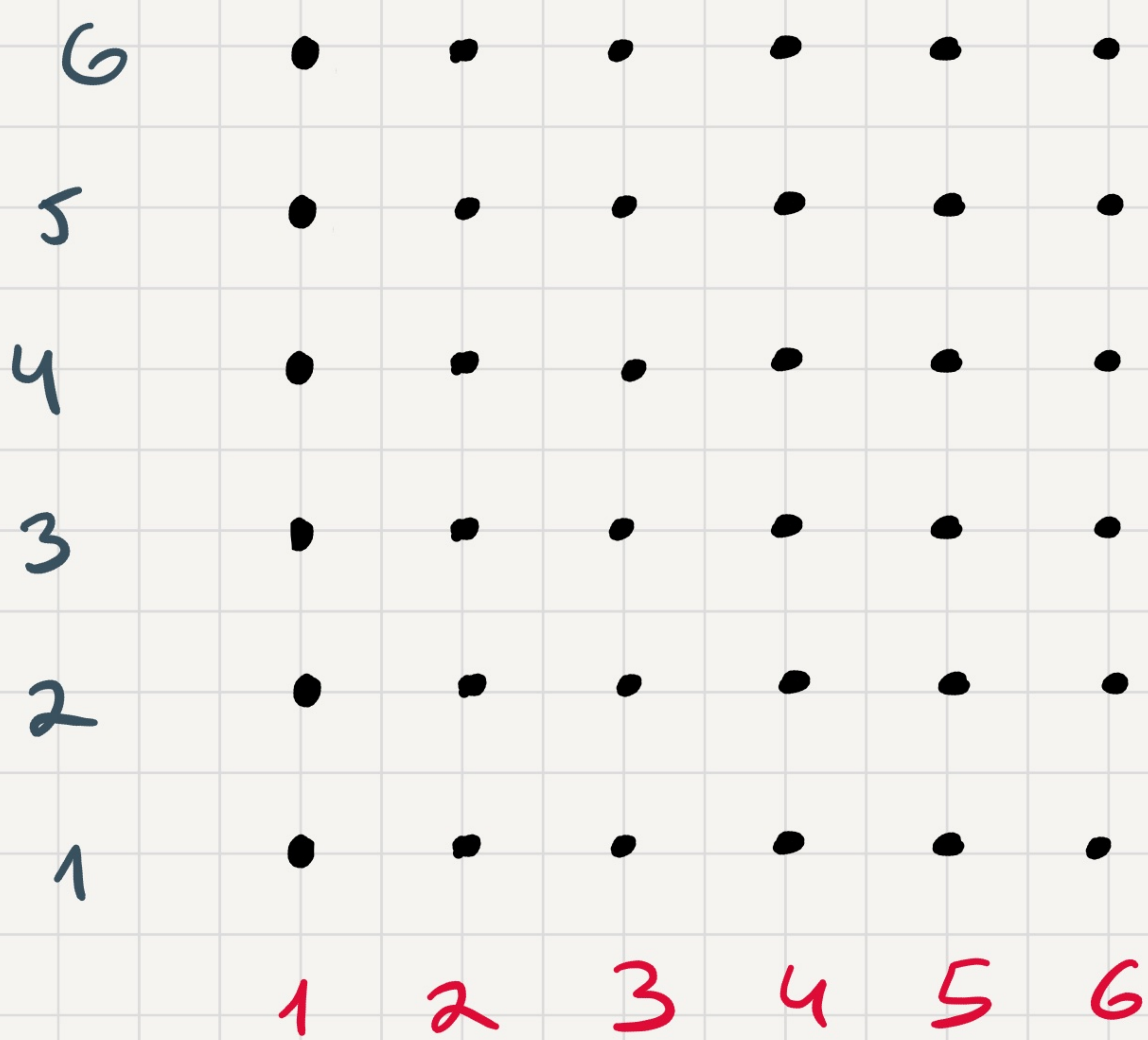
Rolling two dice

X_1 – result of first die

X_2 – " " second die

$X = X_1 + X_2$





Pr. of each outcome $= 1/36$.

$$\mathbb{E}[X] = \mathbb{E}[X_1 + X_2] = \mathbb{E}[X_1] + \mathbb{E}[X_2] = \frac{1+2+\dots+6}{6} + \frac{1+\dots+6}{6}$$

$$= 3.5 + 3.5 = 7$$

Rolling 100 dice

X - total sum of dice.

Q: Calculate $\mathbb{E}[X]$.

Rolling 100 dice

X – total sum of dice.

Q: Calculate $\mathbb{E}[X]$.

X_1 – result of first die

X_2 – " " second die

$\vdots$

X_{100} – " " 100th die

$$X = X_1 + X_2 + \dots + X_{100}$$

$$\mathbb{E}[X] = \mathbb{E}[X_1 + \dots + X_{100}] = \mathbb{E}[X_1] + \dots + \mathbb{E}[X_{100}]$$

$$= 3.5 \cdot 100 = 350.$$

Indicator Random Variable

Defn: Let A be an event. The r.v. X defined as

$$X(\omega) = \begin{cases} 1, & \omega \in A \\ 0, & \text{otherwise} \end{cases}$$

is called the indicator of A.

Usually we denote X by 1_A.

Indicator Random Variable

Def'n: Let A be an event. The r.v. X defined as

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is called the indicator of A.

Usually we denote X by $\mathbb{1}_A$.

$$\Pr[X=1] = \Pr[A], \qquad \Pr[X=0] = 1 - \Pr[A]$$

$$\mathbb{E}[X] = 1 \cdot \Pr[X=1] + 0 \cdot \Pr[X=0] = \Pr[A].$$

Recap

Binomial Distribution

Random Experiment: Flip n biased coins with heads prob. p.

Random Variable: X – number of heads $\quad X \sim Bin(n,p)$

$$\Omega = \{HHH\cdots H,\ HHH\cdots HT,\ \ldots,\ TTT\cdots T\}$$

$$\Pr[X=i] = (\#\text{ of sequences with } i \text{ heads}) \cdot p^i \cdot (1-p)^{n-i}$$

$$= \binom{n}{i} \cdot p^i (1-p)^{n-i}$$

$$\mathbb{E}[X] = ?$$

Binomial Distribution

Random Experiment: Flip n biased coins with heads prob. p.

Random Variable: X – number of heads $\quad X \sim Bin(n,p)$

$$\Omega = \{HHH\cdots H, HHH\cdots HT, \ldots, TTT\cdots T\}$$

Define r.v.s $X_1, X_2, \ldots, X_n$

$$X_i = \begin{cases} 1, & i\text{'th coin flip is heads} \\ 0, & \text{otherwise} \end{cases}$$

$$X = X_1 + \ldots + X_n$$

Binomial Distribution

Random Experiment: Flip n biased coins with heads prob. p.

Random Variable: X – number of heads $X \sim Bin(n,p)$

$$\Omega = \{HHH\cdots H, HHH\cdots HT, \ldots, TTT\cdots T\}$$

Define r.v.s $X_1, X_2, \ldots, X_n$

$$X_i = \begin{cases} 1, & i\text{'th coin flip is heads} \\ 0, & \text{otherwise} \end{cases}$$

$$X = X_1 + \ldots + X_n$$

$$\mathbb{E} X = \mathbb{E} X_1 + \ldots + \mathbb{E} X_n = n \cdot \mathbb{E} X_1 = n \cdot p.$$

Using Linearity of Expectation

Handout assignment at random to n students

X = number of students who got their own assignment.

$\mathbb{E}[X] = ?$

Using Linearity of Expectation

Handout assignment at random to n students

X = number of students who got their own assignment.

$X = X_1 + \dots + X_n$

$$X_i = \begin{cases} 1, & i\text{th student got their assignment} \\ 0, & \text{o.w.} \end{cases}$$

$$\mathbb{E}[X] = \mathbb{E}[X_1] + \dots + \mathbb{E}[X_n]$$

$\forall i \quad \mathbb{E}[X_i] = \frac{1}{n}$

Thus, $\mathbb{E}[X] = 1$

n balls in n bins

X - number of empty bins.

$\mathbb{E}[X] = ?$

n balls in n bins

X - number of empty bins.

X_i - indicator for bin i being empty.

$X = X_1 + \dots + X_n$

$$\Pr[X_1 = 1] = \left(1 - \frac{1}{n}\right)^n \xrightarrow[n \to \infty]{} \frac{1}{e}$$

$$\mathbb{E}[X] = n \cdot \left(1 - \frac{1}{n}\right)^n$$

Summary

- **Joint distribution** for r.v.s X and Y

$$\{(a, b, \Pr[X=a, Y=b]) : a \in \text{range}(X), b \in \text{range}(Y)\}$$

- **Marginal** for X: $\Pr[X=a] = \sum_{b \in \text{range}(Y)} \Pr[X=a, Y=b]$

- **Marginal** for Y $\quad \Pr[Y=b] = \sum_{a \in \text{range}(X)} \Pr[X=a, Y=b]$

- X and Y are **independent** r.v.s if for all a, b

$$\Pr[X=a, Y=b] = \Pr[X=a]\Pr[Y=b]$$

- **Linearity of expectation**: for any two r.v.s X, Y

$$\mathbb{E}[X+Y] = \mathbb{E}[X] + \mathbb{E}[Y].$$

- X is an **indicator** r.v. for event A if $X(\omega) = \begin{cases} 1, & \omega \in A \\ 0, & \text{o.w.} \end{cases}$