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CS70: Lecture 2. Outline.

Today: Proofs!!!

1. By Example.
2. Direct. (Prove $P \implies Q$.)
3. by Contraposition (Prove $P \implies Q$)
4. by Contradiction (Prove P.)
5. by Cases

If time: discuss induction.

Last time: Existential statement.

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Homework.

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A number x is odd if and only if $x = 2k + 1$

Divides.

$a|b$ means

(A) There exists $k \in \mathbb{Z}$, with $a = kb$.

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Therefore Q.

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$n = 121$ Alt Sum: $1 - 2 + 1 = 0$.

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Let D_3 be the 3 digit natural numbers.

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Direct proof of $P \implies Q$:
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Proof: Assume $\neg Q$: d is even. $d = 2k$.

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n is even.

Proof by Contraposition

Thm: For $n \in Z^+$ and $d|n$. If n is odd then d is odd.

$n = kd$ and $n = 2k' + 1$ for integers k, k'.
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Theorem P is true. And proven. □

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Did we prove?

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- Proof assumed no primes *in between* p_k and q.

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The fourth case is the only one possible, so the lemma follows. □

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Question: Which case holds? Don't know!!!

Poll: proof review.

Which of the following are (certainly) true?

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(A),(C),(D)

(B) I don't know.

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Theorem: $3 = 4$

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Proof: Assume $3 = 4$.

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Start with $12 = 12$.

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Divide one side by 3 and the other by 4 to get
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Divide one side by 3 and the other by 4 to get

$$4 = 3.$$

By commutativity theorem holds. $\square$

Be careful.

Theorem: $3 = 4$

Proof: Assume $3 = 4$.

Start with $12 = 12$.

Divide one side by 3 and the other by 4 to get
$4 = 3$.

By commutativity theorem holds. $\square$

Don't assume what you want to prove!

Be really careful!

Theorem: $1 = 2$

Proof:

Be really careful!

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Theorem: $1 = 2$

Proof: For $x = y$, we have

$$(x^2 - xy) = x^2 - y^2$$
$$x(x - y) = (x + y)(x - y)$$

Be really careful!

Theorem: $1 = 2$

Proof: For $x = y$, we have

$$\begin{aligned} (x^2 - xy) &= x^2 - y^2 \\ x(x-y) &= (x+y)(x-y) \\ x &= (x+y) \end{aligned}$$

Be really careful!

Theorem: $1 = 2$

Proof: For $x = y$, we have

$$
\begin{aligned}
(x^2 - xy) &= x^2 - y^2 \\
x(x - y) &= (x + y)(x - y) \\
x &= (x + y) \\
x &= 2x
\end{aligned}
$$

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Poll: What is the problem?

(A) Assumed what you were proving.

(B) No problem. Its fine.

(C) $x - y$ is zero.

(D) Can't multiply by zero in a proof.

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Also: Multiplying inequalities by a negative.

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Dividing by zero is no good. Multiplying by zero is wierdly cool!

Also: Multiplying inequalities by a negative.

$P \implies Q$ does not mean $Q \implies P$.

Summary: Note 2.

Direct Proof:

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Direct Proof:

To Prove: $P \implies Q$.

Summary: Note 2.

Direct Proof:
To Prove: $P \implies Q$. Assume P.

Summary: Note 2.

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To Prove: $P \implies Q$. Assume P. Prove Q.

$a|b$ and $a|c \implies a|(b-c)$.

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To Prove: $P \implies Q$ Assume $\neg Q$.

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n^2 is odd $\implies n$ is odd.

Summary: Note 2.

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To Prove: P

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To Prove: P Assume $\neg P$. Prove False .
$\sqrt{2}$ is rational.

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Either $\sqrt{2}$ and $\sqrt{2}$ worked.

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Don't assume the theorem.

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Careful when proving!

Don't assume the theorem. Divide by zero.

Summary: Note 2.

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To Prove: $P \implies Q$. Assume P. Prove Q.
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By Contraposition:
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n^2 is odd $\implies$ n is odd. $\equiv$ n is even $\implies$ n^2 is even.

By Contradiction:
To Prove: P Assume $\neg P$. Prove False .
$\sqrt{2}$ is rational.
$\sqrt{2} = \frac{a}{b}$ with no common factors....

By Cases: informal.
Universal: show that statement holds in all cases.
Existence: used cases where one is true.
Either $\sqrt{2}$ and $\sqrt{2}$ worked.
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Careful when proving!
Don't assume the theorem. Divide by zero.Watch converse.

Summary: Note 2.

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CS70: Note 3. Induction!

Poll. What's the biggest number?

(A) 100

(B) 101

(C) n+1

(D) infinity.

(E) This is about the "recursive leap of faith."