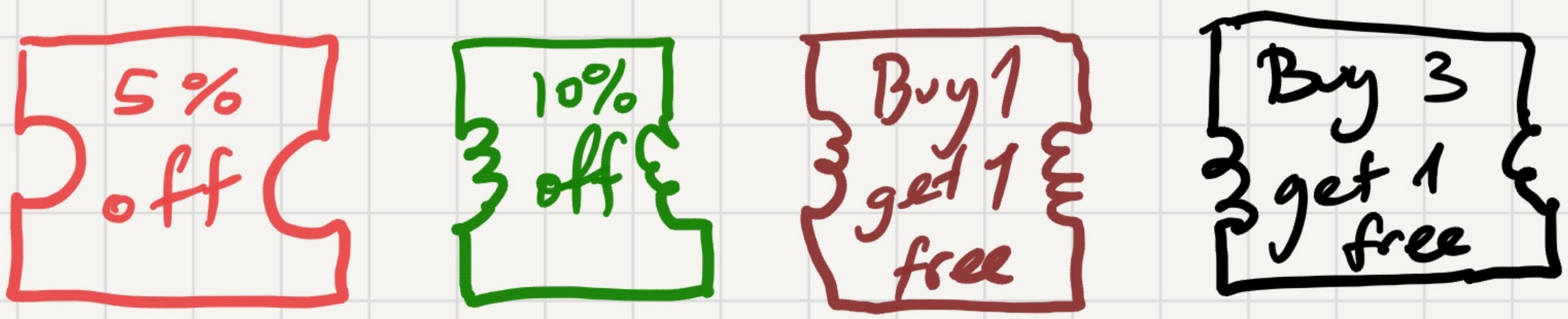
Coupon Collector

Suppose there's a coupon inside every cereal box.



Suppose there are n types of coupons and each box has a random type.

How many boxes do you need to purchase until you get all coupons?

Coupon Collector

Experiment: Get random coupons from $\{1, \dots, n\}$ until you have all types of coupons.

X – # of boxes until you get all n types.

Example: $n=3$

2 2 2 1 1 2 1 3

Idea: Write X as the sum $X_1 + X_2 + X_3$

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$$\underbrace{2}_{X_1} \Big| \underbrace{2\ 2\ 1}_{X_2} \Big| \underbrace{1\ 2\ 1\ 3}_{X_3} \Big|$$

Idea: Write X as the sum $X_1 + X_2 + X_3$

X_1 – # boxes until we get first type

X_2 – # boxes until we get second type after we got first type.

X_3 – # boxes until we get third type after we got second type

Coupon Collector

Experiment: Get random coupons from $\{1, \dots, n\}$ until you have all types of coupons.

X – # of boxes until you get all n types.

Example: $n=3$ $\quad E[X] = ?$

$$2 \mid 2\ 2\ 1 \mid 1\ 2\ 1\ 3 \mid$$
$$X_1 \quad X_2 \quad X_3$$

Idea: Write X as the sum $X_1 + X_2 + X_3$

$1 = X_1$ – # boxes until we get first type

$\mathrm{Geo}(\frac{2}{3}) \leftarrow X_2$ – # boxes until we get second type after we got first type.

$\mathrm{Geo}(\frac{1}{3}) \leftarrow X_3$ – # boxes until we get third type after we got second type

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Example: $n = 3$

$$\underbrace{2}_{X_1} \,\Big|\, \underbrace{2 \; 2 \; 1}_{X_2} \,\Big|\, \underbrace{1 \; 2 \; 1 \; 3}_{X_3} \,\Big|$$

$$\boxed{E}[X] = \boxed{E}[X_1] + \boxed{E}[X_2] + \boxed{E}[X_3]$$

$$= 1 + \frac{3}{2} + 3 = 5.5$$

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General Case:

Write $X = X_1 + X_2 + \dots + X_n$

X_i – # boxes until we get i unique types after we got $i-1$ unique types.

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- $X_1 = 1$
- $X_2 = \begin{cases} 1, & \text{w.p.} \\ 2, & \text{w.p.} \\ \vdots & \end{cases}$
- $X_3 \sim$

$\vdots$

- $X_i \sim$

$X_2 \sim$

General Case:

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X_i - #boxes until we get i unique types after we got $i-1$ unique types.

- $X_1 = 1$
- $X_2 = \begin{cases} 1 & \text{w.p. } \frac{n-1}{n} \\ 2 & \text{w.p. } \frac{1}{n} \cdot \frac{n-1}{n} \\ \vdots & \end{cases}$ $\qquad X_2 \sim Geo\left(\frac{n-1}{n}\right)$
- $X_3 \sim Geo\left(\frac{n-2}{n}\right)$
- $X_i \sim Geo\left(\frac{n-(i-1)}{n}\right)$

$$\mathbb{E}[X] = \mathbb{E}(X_1) + \dots + \mathbb{E}[X_n] =$$

Balls in Bins

Suppose you throw balls in n bins

How many balls do you need to throw until all bins are non-empty?

Functions of Random Variables

Suppose X is a r.v. and $g: \mathbb{R} \to \mathbb{R}$

Then $Y = g(X)$ is also a r.v.

$$Y(\omega) = g(X(\omega)).$$

Q: Calculate $\mathbb{E}[Y]$

Functions of Random Variables

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Q: Calculate $\mathbb{E}[Y]$

Method 1: Compute the dist of Y from the dist of X

Method 2: "LOTUS"

$$\mathbb{E}[Y] = \sum_{\omega \in \Omega} g(X(\omega)) \cdot \Pr[\omega]$$

$$= \sum_{x} \sum_{\omega : X(\omega) = x} g(x) \cdot \Pr[\omega]$$

$$= \sum_{x} g(x) \cdot \sum_{\omega : X(\omega) = x} \Pr[\omega] = \sum_{x} g(x) \cdot \Pr[X = x]$$

Variance

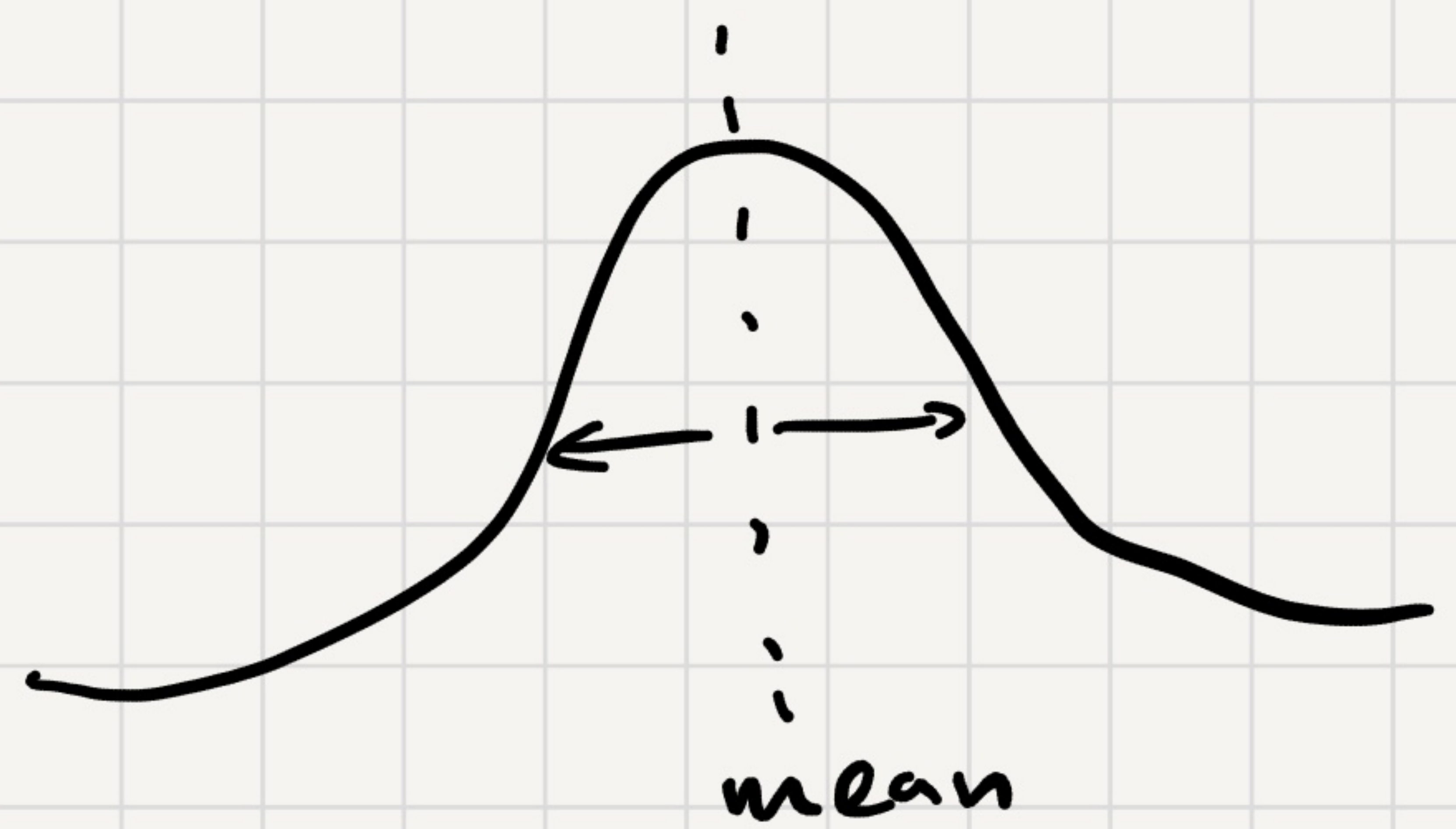
Suppose we have some r.v. X

$\mathbb{E}[X]$ - the average value X would get if we run the experiment many times.

But if we run the experiment only once, we still would like a guarantee on how close X is to its mean.

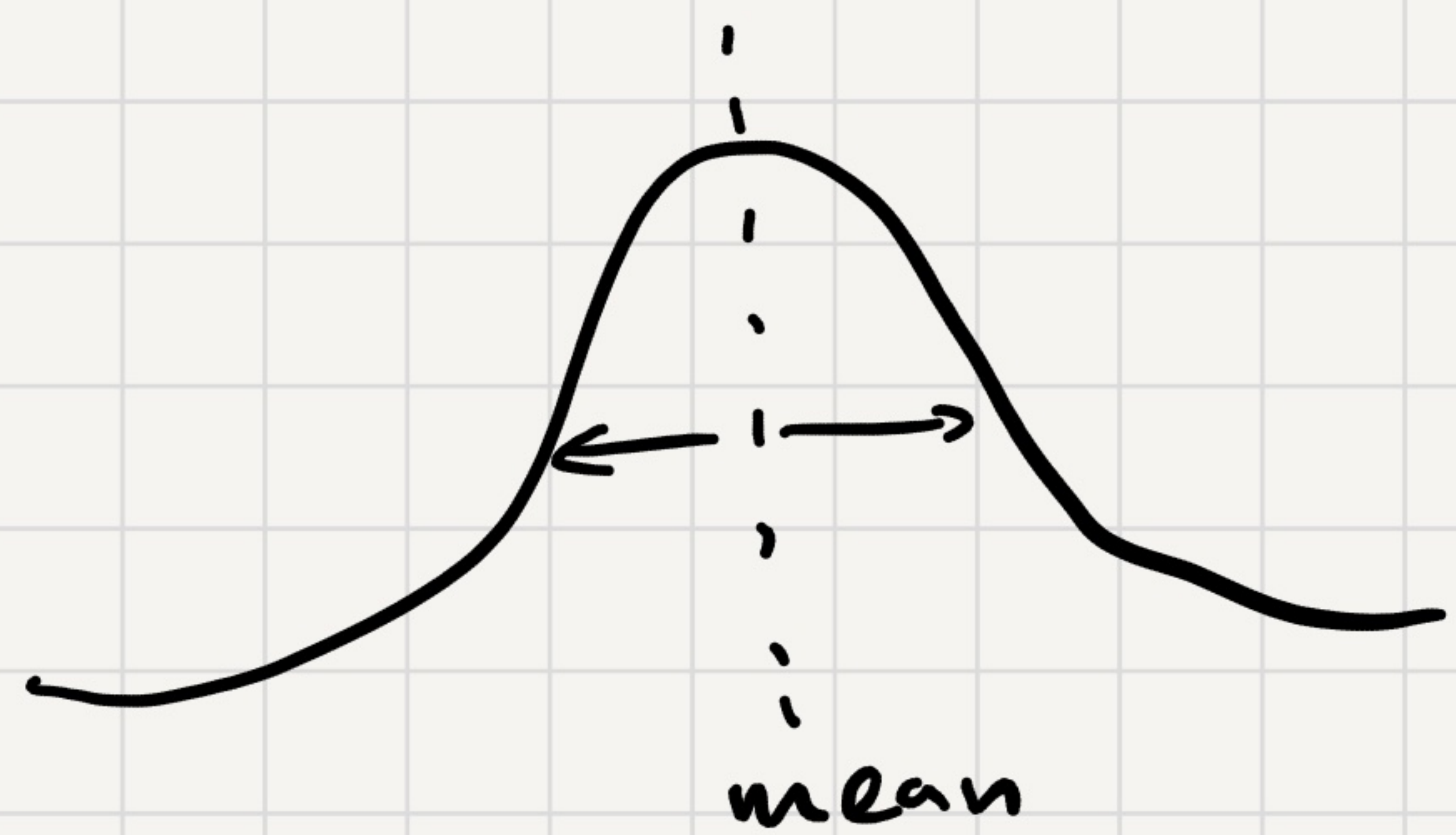
Variance

The variance measures the deviation of X from its mean.



Variance

The variance measures the expected deviation squared of X from its mean.



Def'n: The variance of a r.v. X is defined as

$$\mathrm{Var}(X) = \mathbb{E}\left[(X - \mathbb{E}X)^2\right]$$

Measures the expected deviation squared.

$$\sigma(X) = \mathrm{stdev}(X) = \sqrt{\mathrm{Var}(X)}$$

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Fact: $\mathrm{Var}(X) = \mathbb{E}[X^2] - \mathbb{E}[X]^2.$

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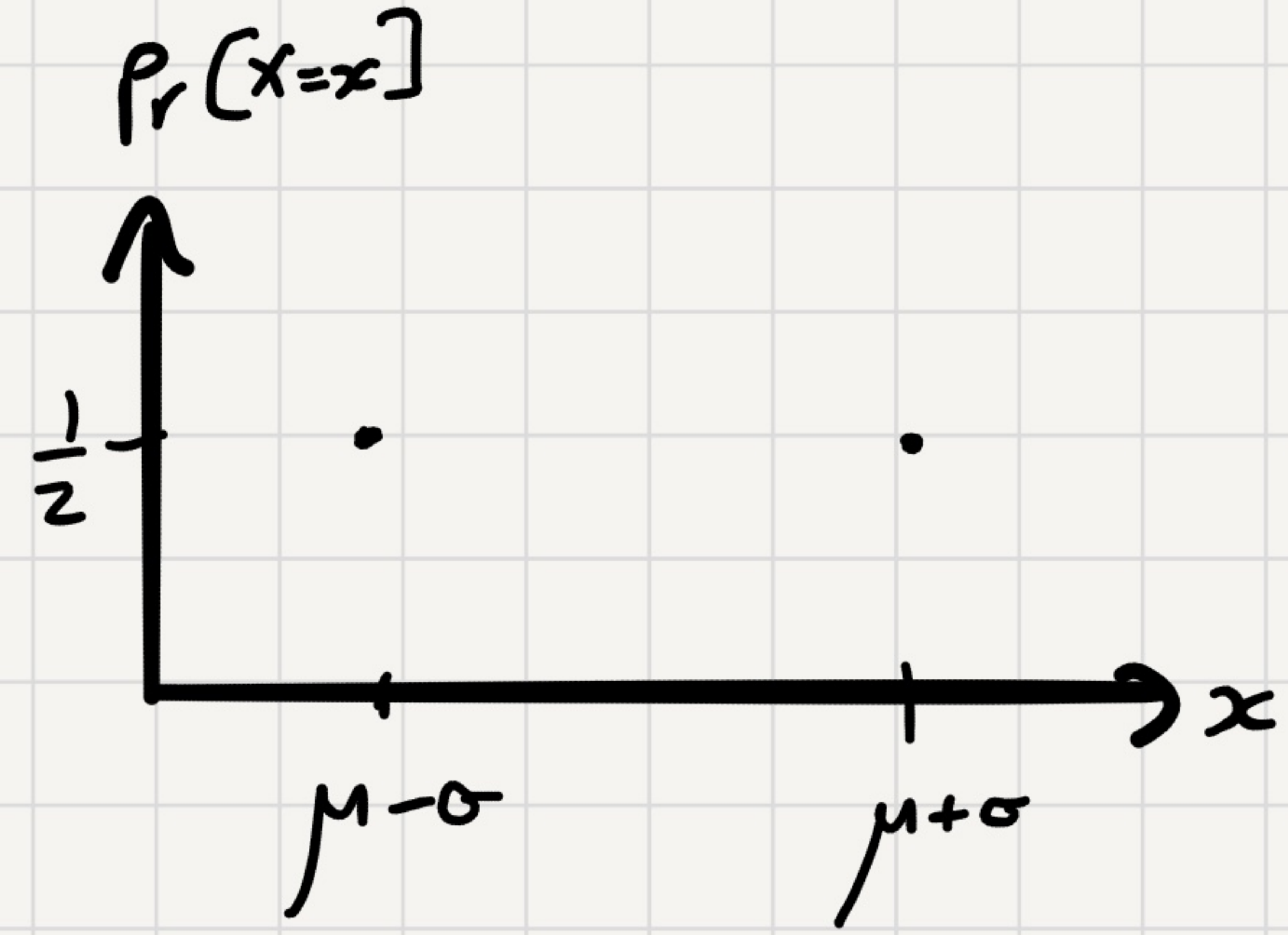
Fact: $\mathrm{Var}(X) = \mathbb{E}[X^2] - \mathbb{E}[X]^2.$

Proof: Let $\mu = \mathbb{E}[X]$.

$$\begin{aligned}
\mathrm{Var}(X) &= \mathbb{E}\left[(X-\mu)^2\right] \\
&= \mathbb{E}\left[X^2 - 2\mu X + \mu^2\right] \\
&= \mathbb{E}[X^2] - 2\mu \underbrace{\mathbb{E}[X]}_{\mu} + \mu^2 \\
&= \mathbb{E}[X^2] - \mu^2 = \mathbb{E}[X^2] - \mathbb{E}[X]^2.
\end{aligned}$$

Simple Example

$$X = \begin{cases} \mu+\sigma & \text{w.p. } \frac{1}{2} \\ \mu-\sigma & \text{w.p. } \frac{1}{2} \end{cases}$$

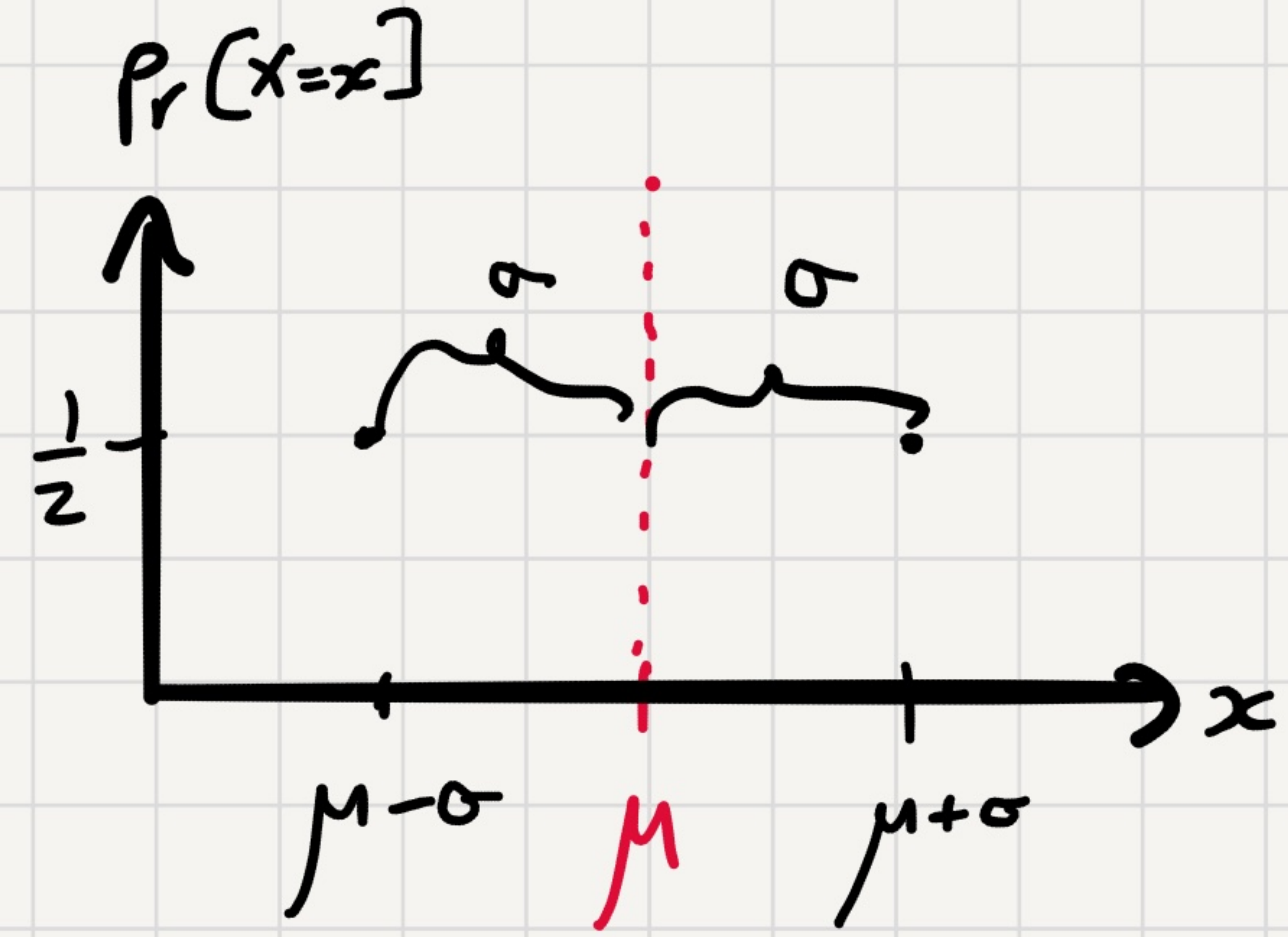


$$\mathbb{E}X = ?$$

$$Var(X) = \mathbb{E}[(X-\mathbb{E}X)^2] = ?$$

Simple Example

$$X = \begin{cases} \mu + \sigma & \text{w.p. } \frac{1}{2} \\ \mu - \sigma & \text{w.p. } \frac{1}{2} \end{cases}$$



$$\mathbb{E}X = \frac{(\mu+\sigma)+(\mu-\sigma)}{2} = \mu$$

$$\mathrm{Var}(X) = \mathbb{E}[(X-\mathbb{E}X)^2] = \frac{1}{2}\cdot(\mu+\sigma-\mu)^2 + \frac{1}{2}(\mu-\sigma-\mu)^2$$

$$= \frac{1}{2}\sigma^2 + \frac{1}{2}\sigma^2 = \sigma^2$$

Facts: For any $c \in \mathbb{R}$, r.v. X

1. $Var(c \cdot X) = c^2 \cdot Var(X)$. Scales by c^2
2. $Var(c + X) = Var(X)$. shifts center

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Proof:

1. $Var(c \cdot X) = \mathbb{E}[(cX)^2] - \mathbb{E}[cX]^2$

$= c^2 \cdot \mathbb{E}[X^2] - c^2 \cdot \mathbb{E}[X]^2 = c^2 \cdot Var(X)$.

2. $Var(c+X) = \mathbb{E}\left[\left(c+X - \mathbb{E}(c+X)\right)^2\right]$

$= \mathbb{E}\left[\left(c+X - c - \mathbb{E}X\right)^2\right]$

$= \mathbb{E}\left[(X - \mathbb{E}X)^2\right] = Var(X)$.

Facts: For any $c \in \mathbb{R}$, r.v. X

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If X and Y are r.v.s.
what's $Var(X+Y) = ?$

Recap

Independent Random Variables

Def'n: We say that two r.v.s X and Y are independent if

$$\Pr[X=a, Y=b] = \Pr[X=a] \cdot \Pr[Y=b]$$

for all $a \in \text{range}(X)$
$b \in \text{range}(Y)$.

Fact: If X & Y are indep.

$$\Pr[X=a \mid Y=b] = \Pr[X=a]$$

Theorem: If X and Y are independent r.v.s

then $$\mathbb{E}[X \cdot Y] = \mathbb{E}[X] \cdot \mathbb{E}[Y]$$

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then $$\mathbb{E}[X \cdot Y] = \mathbb{E}[X] \cdot \mathbb{E}[Y]$$

Proof:

$$\begin{aligned}
\mathbb{E}[X \cdot Y] &= \sum_{a,b} a \cdot b \cdot \Pr[X=a, Y=b] \\
&= \sum_{a,b} a \cdot b \cdot \Pr[X=a] \cdot \Pr[Y=b] \\
&= \left(\sum_{a} a \cdot \Pr[X=a]\right) \cdot \left(\sum_{b} b \cdot \Pr[Y=b]\right) \\
&= \mathbb{E}[X] \cdot \mathbb{E}[Y].
\end{aligned}$$

Theorem: If X and Y are independent r.v.s, then

$$\mathrm{Var}(X+Y) = \mathrm{Var}(X) + \mathrm{Var}(Y).$$

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Theorem: If X and Y are independent r.v.s, then

$$Var(X+Y) = Var(X) + Var(Y).$$

Proof: $\mathbb{E}[(X+Y)^2] = \mathbb{E}[X^2] + 2\mathbb{E}[XY] + \mathbb{E}[Y^2]$

$$(\mathbb{E}[X+Y])^2 = (\mathbb{E}X + \mathbb{E}Y)^2$$

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$$\mathrm{Var}(X+Y) = \mathbb{E}[X^2] - (\mathbb{E}X)^2 + 2\,\mathbb{E}[XY] - 2(\mathbb{E}X)(\mathbb{E}Y) + \mathbb{E}[Y^2] - (\mathbb{E}Y)^2$$

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$$= \mathrm{Var}(X) + 0 + \mathrm{Var}(Y)$$

since X, Y are indep.

Theorem: If $X_1, X_2, \ldots, X_n$ are pairwise independent r.v.s then

$$\mathrm{Var}(X_1 + \ldots + X_n) = \mathrm{Var}(X_1) + \ldots + \mathrm{Var}(X_n)$$

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Proof: (Similar)

$$\mathbb{E}[(X_1 + \cdots + X_n)^2] = \sum_{i=1}^{n} \mathbb{E}[X_i^2] + \sum_{i \neq j} \mathbb{E}[X_i X_j]$$

$$\mathbb{E}[X_1 + \cdots + X_n]^2 = \sum_{i=1}^{n} \mathbb{E}[X_i]^2 + \sum_{i \neq j} \mathbb{E}[X_i]\mathbb{E}[X_j]$$

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$$= \sum_{i=1}^{n} Var(X_i) + 0.$$

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1. $\mathrm{Var}(c \cdot X) = c^2 \cdot \mathrm{Var}(X)$. Scales by c^2
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Q: If X and Y are r.v.s. what's $\mathrm{Var}(X+Y) = ?$

A: It depends.

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A: It depends.

Example 1: X, Y ind.

Example 2: X, Y r.v.s $Y = X$

Example 3: X, Y r.v.s $Y = -X$

Variance of a Binomial Distribution

$X \sim \mathrm{Bin}(n,p)$ models # of heads in n coin tosses w. heads prob. p.

$$\Pr[X=i] = \binom{n}{i} \cdot p^i \cdot (1-p)^{n-i}$$

$$\mathrm{Var}(X) = ?$$

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$X = X_1 + \dots + X_n$ $\quad$ X_i = indicator that ith flip is heads.

$$\mathbb{E}X = \mathbb{E}X_1 + \dots + \mathbb{E}X_n = n \cdot p.$$

$Var(X) =$?

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$$\mathrm{Var}(X) = \mathrm{Var}(X_1 + \dots + X_n) \underset{\uparrow}{=} \mathrm{Var}(X_1) + \dots + \mathrm{Var}(X_n)$$

$X_1, \dots, X_n$ are pairwise indep.

$$\mathrm{Var}(X_1) = ?$$

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$$Var(X) = Var(X_1 + \dots + X_n) = Var(X_1) + \dots + Var(X_n)$$

$$Var(X_i) = \mathbb{E}[X_i^2] - \mathbb{E}[X_i]^2 = p - p^2 = p \cdot (1-p)$$

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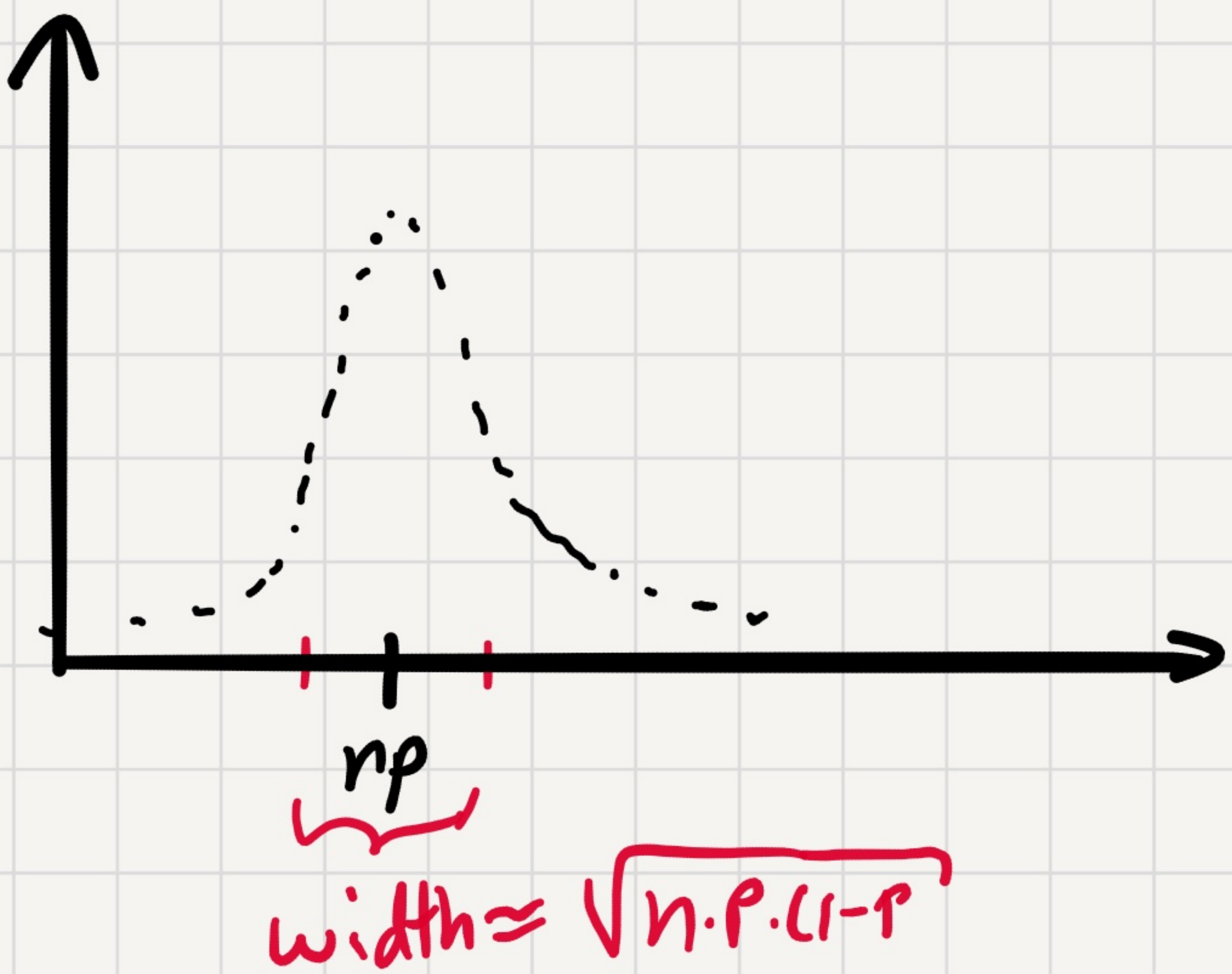
$$\Pr[X=i] = \binom{n}{i} \cdot p^i \cdot (1-p)^{n-i}$$

$$\mathbb{E}X = np$$

Scales linearly with n $\rightarrow$ $\mathrm{Var}(X) = n \cdot p \cdot (1-p) \le \frac{n}{4}$

$$\mathrm{stdev}(X) = \sqrt{n \cdot p \cdot (1-p)} \le \frac{\sqrt{n}}{2}$$

Scales like $\sqrt{n}$



Number of Fixed Points - Variance

Handout assignment at random to n students

X = number of students who got their own assignment.

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$$\mathbb{E}[X^2] = \mathbb{E}\left[(X_1 + \dots + X_n)^2\right] = \mathbb{E}\left[\sum_{i=1}^{n} X_i^2 + \sum_{i \neq j} X_i X_j\right]$$

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$$= \sum_{i=1}^n \mathbb{E}[X_i^2] + \sum_{i \neq j} \mathbb{E}[X_i X_j]$$

$$\mathbb{E}[X_i X_j] = \Pr[\pi(i) = i, \pi(j) = j] = \frac{1}{n} \cdot \frac{1}{n-1}$$

are X_i, X_j indep?

$$\mathbb{E}[X^2] = \sum_{i=1}^n \mathbb{E}[X_i^2] + \sum_{i \neq j} \frac{1}{n} \cdot \frac{1}{n-1} = n \cdot \frac{1}{n} + n \cdot (n-1) \frac{1}{n} \frac{1}{n-1} = 2.$$

Poll: What's true

1. X_i & X_j are ind.

2. $\mathbb{E}[X_i X_j] = \Pr[X_i X_j = 1]$

3. $\mathbb{E}[X_i X_j] = \Pr[X_i = 1 \cap X_j = 1]$

4. $X_i^2 = X_i$

Covariance

If X and Y are r.v.s their co-variance is

$$\mathrm{Cov}(X, Y) = \mathbb{E}\left[(X - \mathbb{E}X)(Y - \mathbb{E}Y)\right]$$

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Fact: $\operatorname{Cov}(X,Y) = \mathbb{E}[XY] - (\mathbb{E}X)\cdot(\mathbb{E}Y)$

Proof: let $\mu_x = \mathbb{E}X$ $\quad \mu_y = \mathbb{E}Y$

$$\begin{aligned}
\operatorname{cov}(X,Y) &= \mathbb{E}\left[(X-\mu_x)(Y-\mu_Y)\right] \\
&= \mathbb{E}[XY] - \mu_x\mathbb{E}[Y] - \mu_Y\mathbb{E}[X] + \mu_x\mu_y \\
&= \mathbb{E}[XY] - \mu_x\mu_y = \mathbb{E}[XY] - (\mathbb{E}X)(\mathbb{E}Y).
\end{aligned}$$

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If X and Y are r.v.s their co-variance is

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Fact: $\mathrm{Cov}(X,Y) = \mathbb{E}[XY] - (\mathbb{E}X)\cdot(\mathbb{E}Y)$

Corollary: If X and Y are independent then

$$\mathrm{Cov}(X,Y) = 0.$$

Recap

Theorem: If X and Y are independent r.v.s, then

$$\mathrm{Var}(X+Y) = \mathrm{Var}(X) + \mathrm{Var}(Y).$$

Proof:

$$-\begin{cases} \mathbb{E}[(X+Y)^2] = \mathbb{E}[X^2] + 2\mathbb{E}[XY] + \mathbb{E}[Y^2] \\ (\mathbb{E}[X+Y])^2 = (\mathbb{E}X + \mathbb{E}Y)^2 \\ \quad = (\mathbb{E}X)^2 + 2(\mathbb{E}X)\cdot(\mathbb{E}Y) + (\mathbb{E}Y)^2 \end{cases}$$

$$\mathrm{Var}(X+Y) = \mathbb{E}[X^2] - (\mathbb{E}X)^2 + 2\,\mathbb{E}[XY] - 2(\mathbb{E}X)(\mathbb{E}Y) + \mathbb{E}[Y^2] - (\mathbb{E}Y)^2$$

$$= \mathrm{Var}(X) + 0 + \mathrm{Var}(Y)$$

The only point where we used that X & Y are indep.

Theorem: If X and Y are r.v.s

$$Var(X+Y) = Var(X) + Var(Y) + 2\,cov(X,Y)$$

Proof:

$$-\begin{cases} \mathbb{E}[(X+Y)^2] = \mathbb{E}[X^2] + 2\,\mathbb{E}[XY] + \mathbb{E}[Y^2] \\ (\mathbb{E}[X+Y])^2 = (\mathbb{E}X + \mathbb{E}Y)^2 \end{cases}$$

$$= (\mathbb{E}X)^2 + 2(\mathbb{E}X)\cdot(\mathbb{E}Y) + (\mathbb{E}Y)^2$$

$$Var(X+Y) = \mathbb{E}[X^2] - (\mathbb{E}X)^2 + 2\,\mathbb{E}[XY] - 2(\mathbb{E}X)(\mathbb{E}Y) + \mathbb{E}[Y^2] - (\mathbb{E}Y)^2$$

$$= Var(X) + 2\cdot cov(X,Y) + Var(Y)$$