Lecture 23

Continuous Probability I

Plan for Today

1. Motivating Example - uniform $X \in [0,1]$.
2. Probability of events
3. Continuous Random Variables & Distributions.
4. <u>**Definitions:**</u> Cumulative distribution function
   probability density function.
5. Exponential RVs.

Uniformly at random in $[0,1]$.

Choose a real number X, uniformly at random in $[0,1]$.

Q: What's the probability that $X = 0.3$?

A: 0.

For any x, $\Pr[X = x] = 0$.

What about $\Pr(X \le \frac{1}{2}] = ?$

$\Pr[X \in [0.3, 0.4]] = ?$

$\Pr(X \in [a,b]]) = ?$

Uniformly at random in $[0,1]$.

Choose a real number X, uniformly at random in $[0,1]$.

For any $0 \le a < b \le 1$ the event "$X \in [a,b]$" has prob. $b-a$.

$$\Pr[X \in [0,0.1] \text{ or } X \in [0.8,0.9]]$$

$$= \Pr[X \in [0,0.1]] + \Pr[X \in [0,0.1]] + \Pr[X \in [0.8,0.9]] = 0.2.$$

In General: If $E_1, E_2, E_3, \dots$ are disjoint events

$$\Pr[E_1 \cup E_2 \cup \dots] = \sum_i \Pr[E_i]$$

Uniformly at random in [0,1].

Choose a real number X, uniformly at random in $[0,1]$.

Define
$$F_X(x) = \Pr[X \le x]$$

$$F_X(x) = \begin{cases} 0 & , \text{ if } x < 0 \\ x & , \text{ if } 0 \le x \le 1 \\ 1 & , \text{ if } 1 < x \end{cases}$$

$$\Pr[X \in (a,b]] = \Pr[X \le b] - \Pr[X \le a]$$
$$= F_X(b) - F_X(a).$$

$\Rightarrow F_X$ specifies Prob. of all events

Definition:

Let X be a real valued random variable. The cumulative distribution function (c.d.f.) of X is

$$F_X(x) = \Pr[X \leq x]$$

Notes:

1. A r.v. is continuous if F_X is continuous.
2. The c.d.f. defines the prob. of each interval

$$\Pr[X \in (a,b]] = F_X(b) - F_X(a)$$

3. Hence, the c.d.f defines the prob. of each event (union of intervals).

Discrete vs Continuous Probability

Discrete	Continuous
Finite or Countable Sample Space Ω	Continuous Sample Space.
Start with $\Pr[\omega]$	$\Pr[\omega]$ is typically 0.
Derive prob. of events: $\Pr[E] = \sum_{\omega \in E} \Pr[\omega]$	Start with prob. of events, e.g. intervals, union of intervals.

Uniformly at random in $[0,1]$ - Discrete Approximation

X uniformly random in $[0,1]$

Let n be a large integer.

$Y = \frac{\lceil X \cdot n \rceil}{n}$. In other words, if $\frac{i-1}{n} < X \leq \frac{i}{n}$ then $Y = \frac{i}{n}$.

$$\Pr\left[Y = \frac{i}{n}\right] = \Pr\left[X \in \left(\frac{i-1}{n}, \frac{i}{n}\right]\right] = \frac{1}{n}$$

Y is uniform in $\left\{\frac{1}{n}, \frac{2}{n}, \frac{3}{n}, \dots, 1\right\}$

$$|X - Y| \leq \frac{1}{n}$$

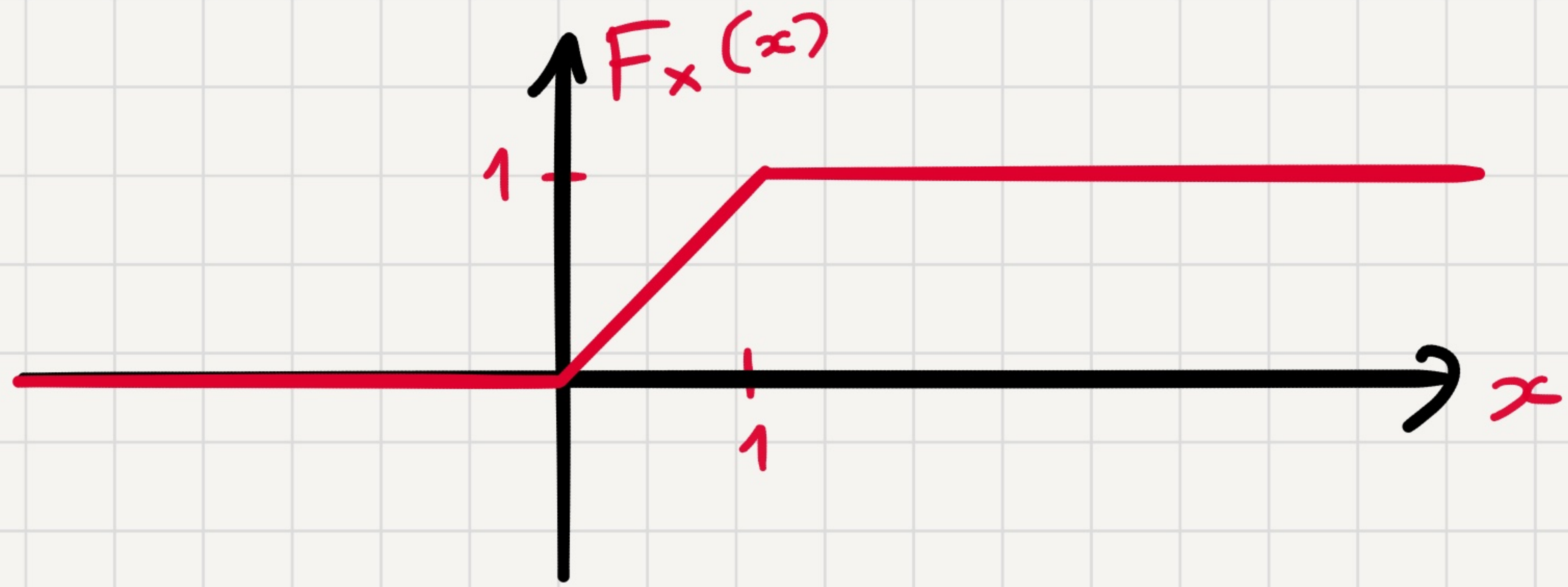
Uniformly at random in [0,1].

Choose a real number X, uniformly at random in $[0,1]$.

Define

$$F_X(x) = \Pr[X \leq x]$$

$$F_X(x) = \begin{cases} 0 & , \text{if } x < 0 \\ x & , \text{if } 0 \leq x \leq 1 \\ 1 & , \text{if } 1 < x \end{cases}$$

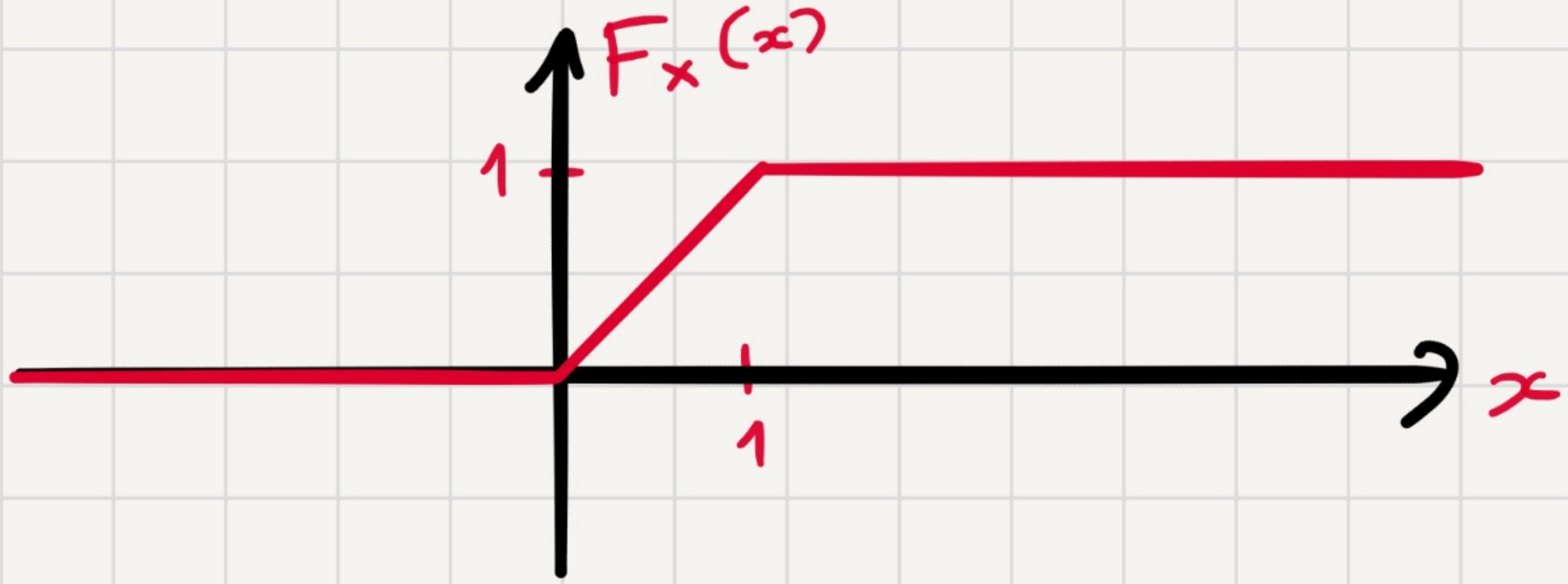


Uniformly at random in [0,1].

Choose a real number X, uniformly at random in $[0,1]$.

Define

$$F_X(x) = Pr[X \le x] \qquad F_X(x) = \begin{cases} 0 & , \text{ if } x < 0 \\ x & , \text{ if } 0 \le x \le 1 \\ 1 & , \text{ if } 1 < x \end{cases}$$



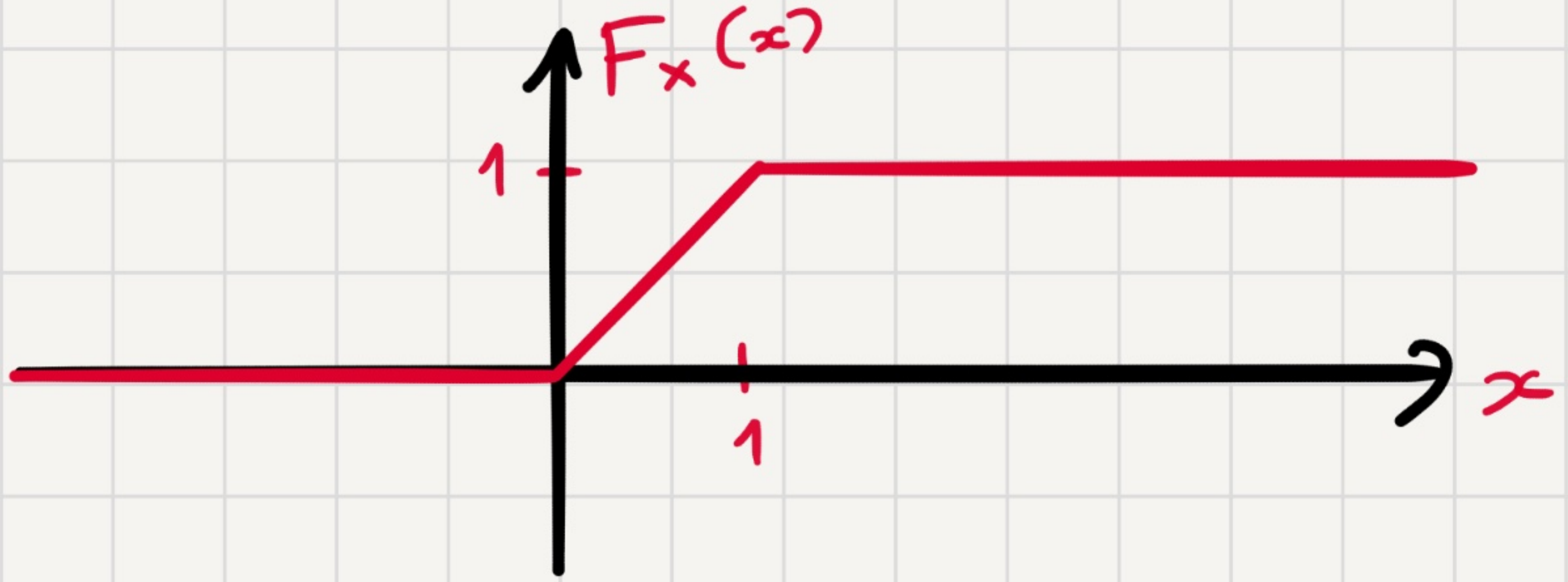
Define $\quad f_X(x) = \lim_{\delta \to 0^+} \frac{Pr[X \in (x, x+\delta)]}{\delta}$

probability density function

Uniformly at random in [0,1].

Choose a real number X, uniformly at random in $[0,1]$.

Define $F_X(x) = Pr[X \leq x]$ $\quad F_X(x) = \begin{cases} 0 & , \text{ if } x < 0 \\ x & , \text{ if } 0 \leq x \leq 1 \\ 1 & , \text{ if } 1 < x \end{cases}$



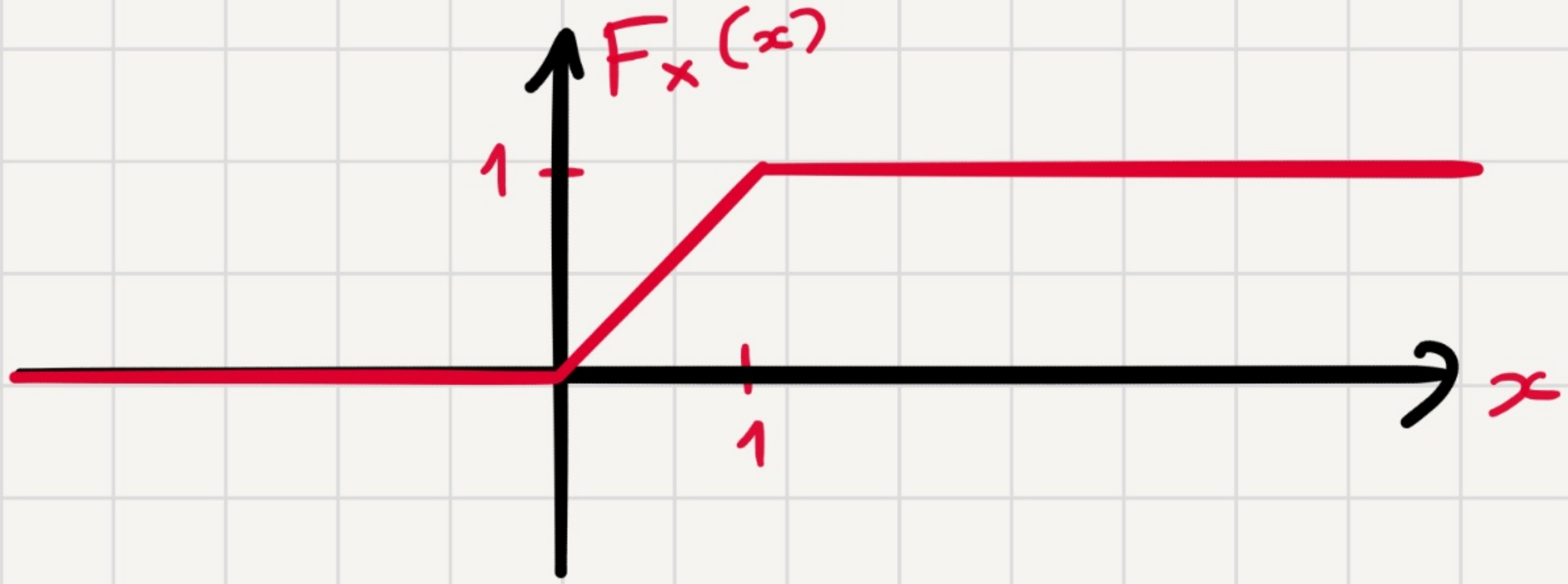
Define $\quad f_X(x) = \lim_{\delta \to 0^+} \dfrac{Pr[X \in [x, x+\delta]]}{\delta}$

then, $\quad f_X(x) = \lim_{\delta \to 0^+} \dfrac{F_X(x+\delta) - F_X(x)}{\delta} = F_X'(x)$

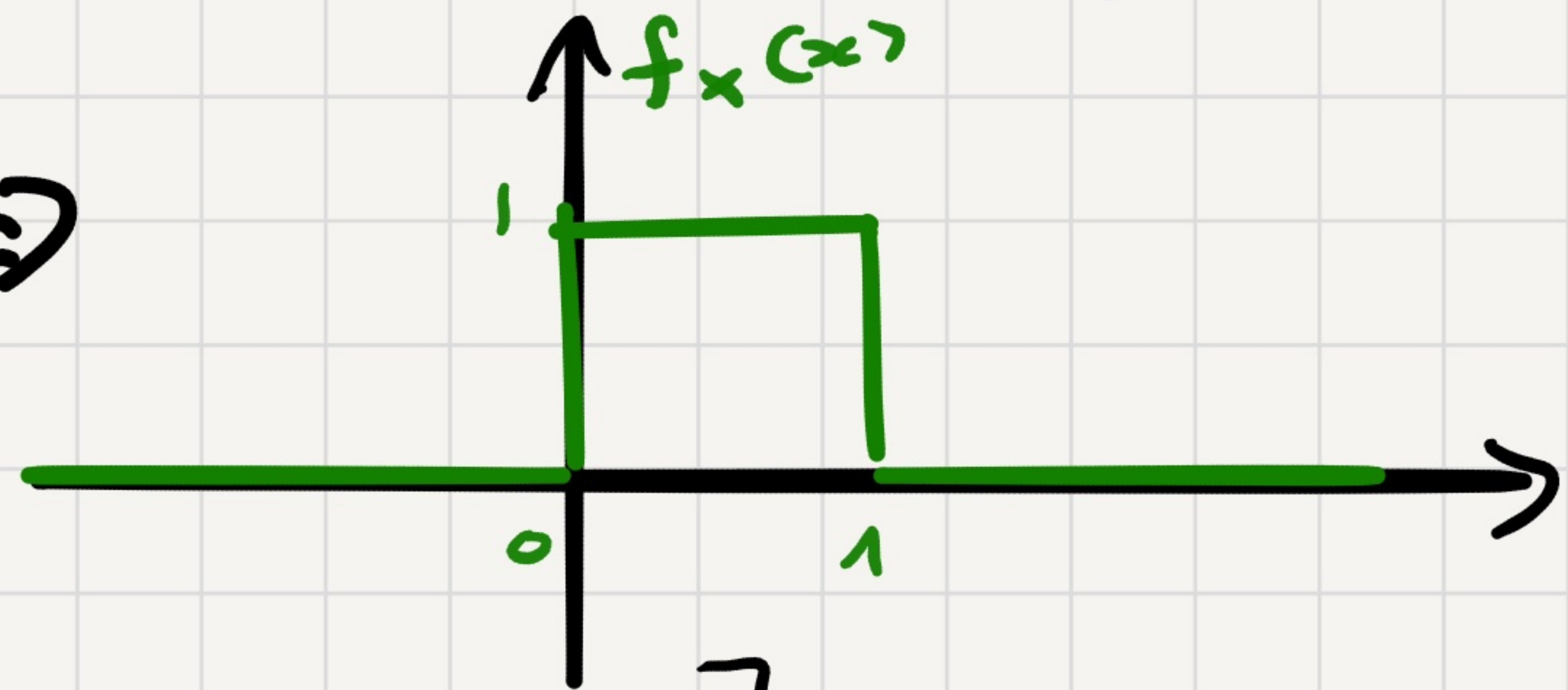
Uniformly at random in [0,1].

Choose a real number X, uniformly at random in $[0,1]$.

Define $F_X(x) = Pr[X \le x]$ $\quad F_X(x) = \begin{cases} 0 & , \text{if } x < 0 \\ x & , \text{if } 0 \le x \le 1 \\ 1 & , \text{if } 1 < x \end{cases}$



$\Rightarrow$



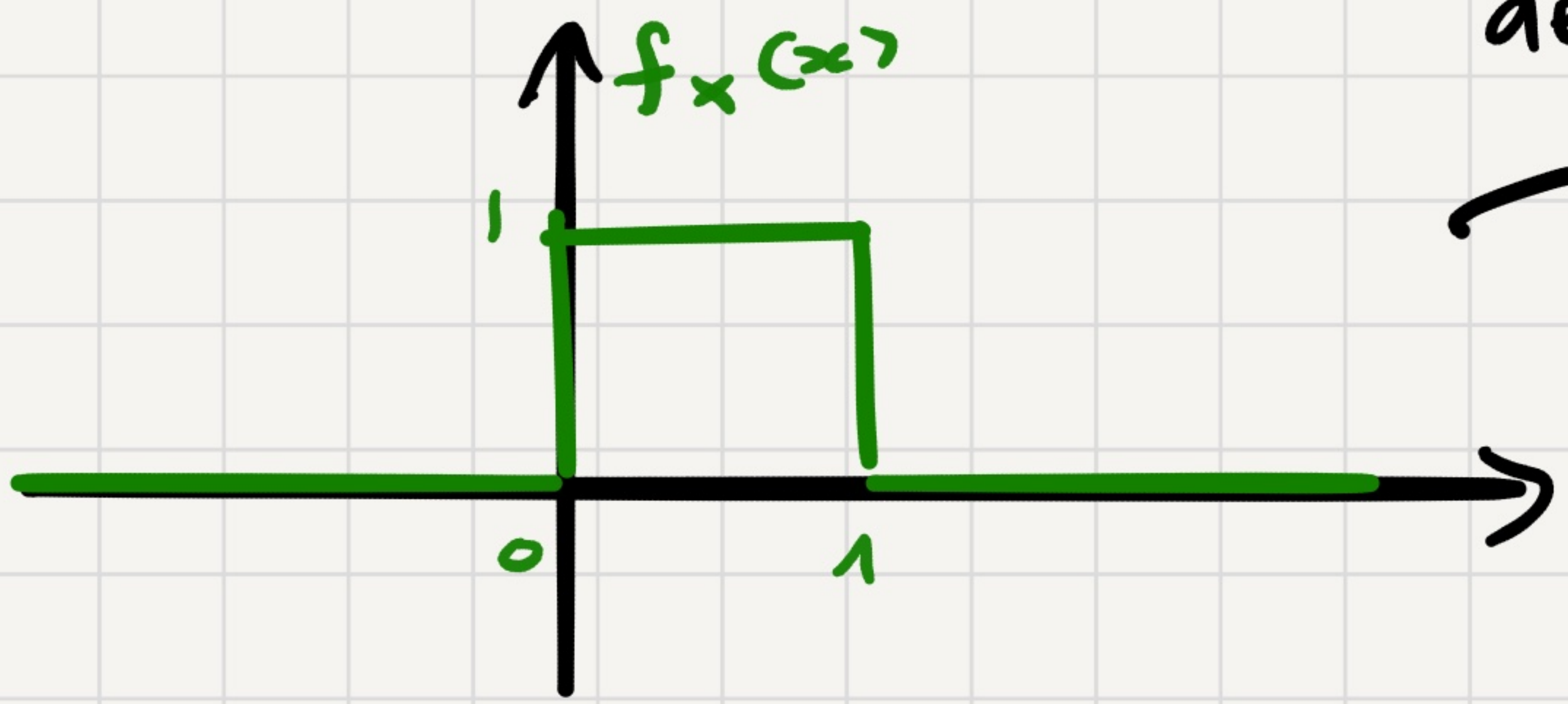
Define $f_X(x) = \lim_{\delta \to 0^+} \dfrac{Pr[X \in [x, x+\delta]]}{\delta}$

then, $f_X(x) = F_X'(x) = \begin{cases} 0 & , \text{if } x < 0 \\ 1 & , \text{if } 0 \le x < 1 \\ 0 & , \text{if } 1 \le x \end{cases}$

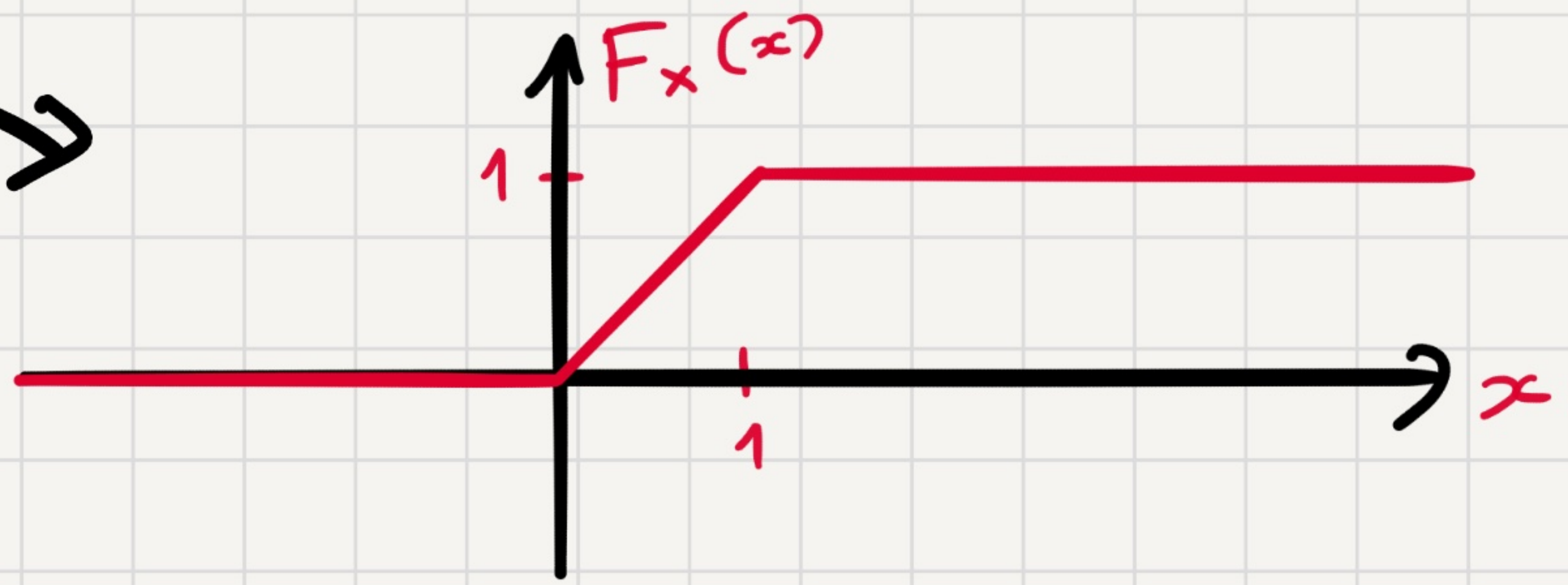
Uniformly at random in [0,1].

Choose a real number X, uniformly at random in [0,1].

Alternative Derivation: Start with the density function and derive the c.d.f.



derive c.d.f.



If $f_X(x) = F_X'(x)$

Then, $F_X(b) - F_X(a) = \int_a^b f_X(x)\,dx.$

$$F_X(a) = \int_{-\infty}^{a} f_X(x)\,dx$$

Definition:

Let X be a real valued random variable

The cumulative distribution function (c.d.f.) of X is

$$F_X(x) = \Pr[X \le x]$$

The probability density function (p.d.f) of X is

$$f_X(x) = \lim_{\delta \to 0^+} \frac{\Pr[X \in [x, x+\delta)]}{\delta}$$

Definition:

Let X be a real valued random variable

The cumulative distribution function (c.d.f.)

of X is

$$F_X(x) = Pr[X \leq x]$$

The probability density function (p.d.f) of X

is

$$f_X(x) = \lim_{\delta \to 0^+} \frac{Pr[X \in [x, x+\delta]]}{\delta}$$

$$f_X(x) = F'_X(x)$$

$$F_X(b) - F_X(a) = \int_a^b f_X(x)\, dx$$

Definition:

Let X be a real valued random variable

The cumulative distribution function (c.d.f.) of X is

$$F_X(x) = \Pr[X \leq x]$$

The probability density function (p.d.f) of X is

$$f_X(x) = \lim_{\delta \to 0^+} \frac{\Pr[X \in [x, x+\delta]]}{\delta}$$

For small dx,

$$\Pr[X \in [x, x+dx]] \approx f(x)\,dx$$

Properties of Density Functions

1. $f_X(x)$ is non-negative

2. $\int_{-\infty}^{\infty} f_X(x)\, dx = \boxed{}$

Q: Can $f_X(x)$ be larger than 1?

Alternative Definition

Let X be a real valued random variable.

The probability density function (p.d.f) of X is a non-negative function $f(x)$ s.t. $\int_{-\infty}^{\infty} f(x)\,dx = 1$ and for all $a<b$:

$$\Pr[a \leq X \leq b] = \int_a^b f(x)\,dx.$$

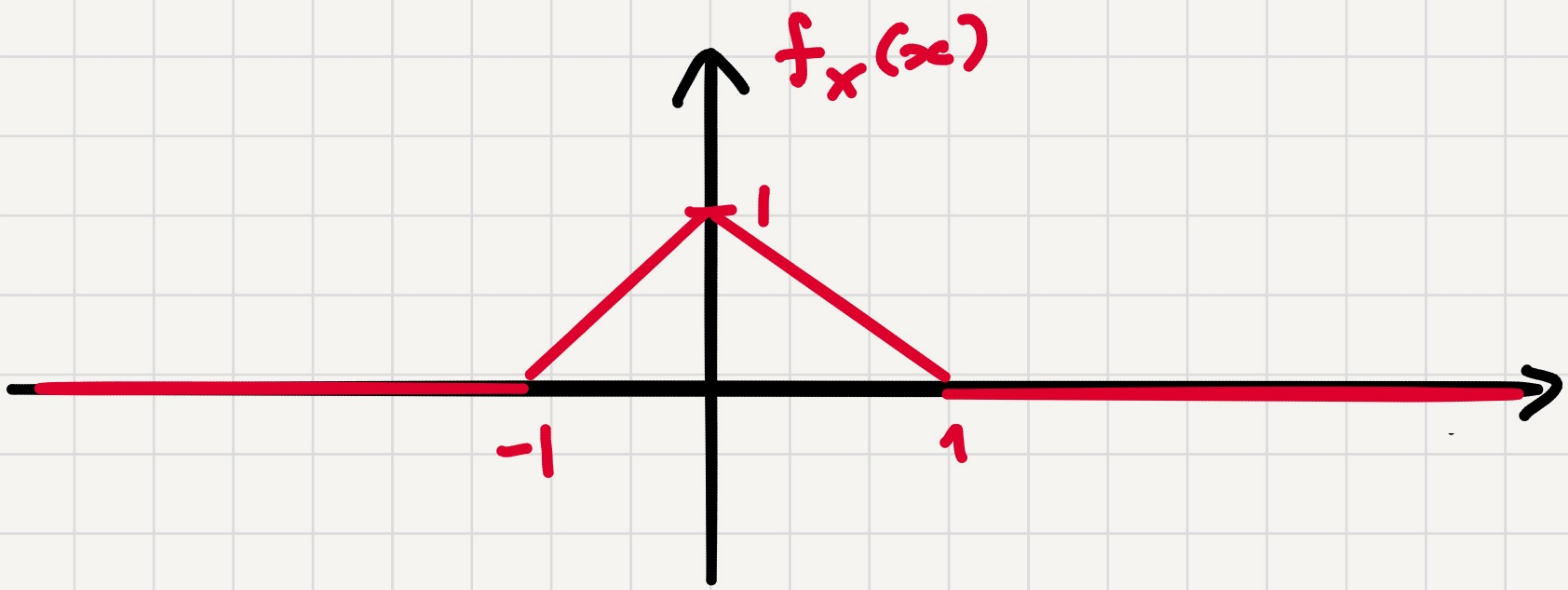
Uniform in $[0, \ell]$

$$F_X(x) = \boxed{\quad\quad\quad} \quad \text{for } 0 \le x \le \ell$$

$$f_X(x) = \begin{cases} \boxed{\quad\quad\quad}, & \text{for } 0 \le x \le \ell \\ \boxed{\quad\quad}, & \text{o.w.} \end{cases}$$

Example #2

Let X be a continuous r.v. with p.d.f.



$$f_X(x) = \begin{cases} 0 & \text{if } x < -1 \\ x+1 & \text{if } -1 \leq x < 0 \\ 1-x & \text{if } 0 \leq x < 1 \\ 0 & \text{if } 1 \leq x \end{cases}$$

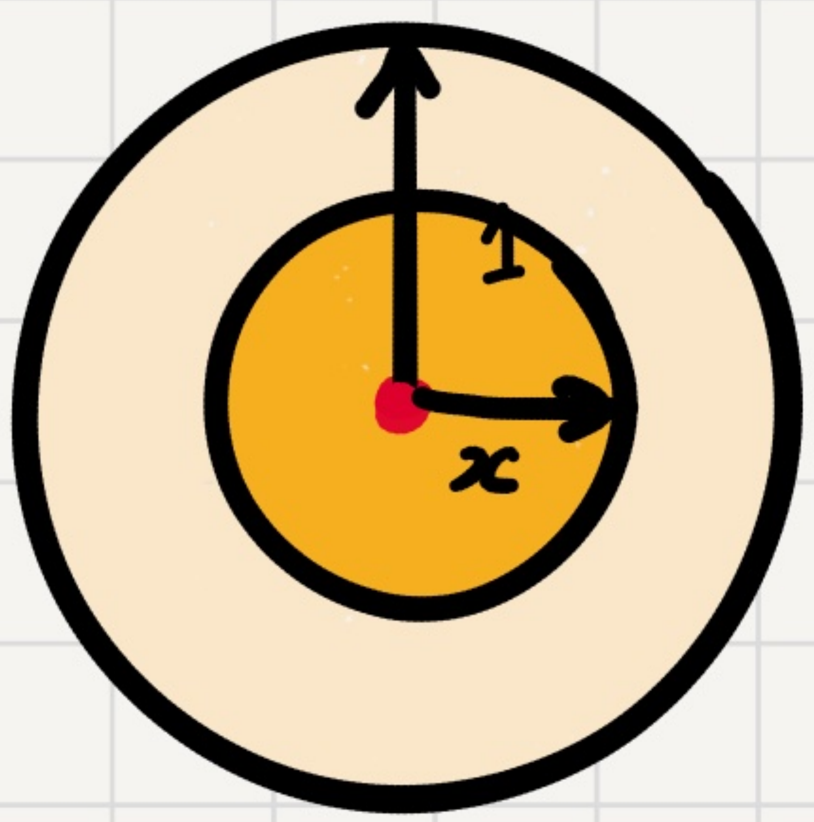
$$\int_{-\infty}^{\infty} f_X(x)\, dx = \text{area under curve} = 1.$$

Example #3

You throw a dart at a dartboard at random and you hit it. X - distance from center.

What's the distribution of X?

$$\Pr[X \le x] = \frac{\text{Area of small circle}}{\text{Area of dart board}}$$



$$= \frac{\pi x^2}{\pi \cdot 1^2} = x^2$$

$$F_X(x) = \begin{cases} 0, & x < 0 \\ x^2, & 0 \le x < 1 \\ 1, & 1 \le x \end{cases}$$

$f_X(x) = ?$

Exponential Random Variable

This is the analog to the Geometric R.V.

Recall: $X \sim Geo(p)$

measures how many trials until we get H

where $Pr[H] = p$.

$$Pr[X = i] = (1-p)^{i-1} \cdot p.$$

$$Pr[X > i] = (1-p)^{i}$$

Exponential Random Variable

This is the analog to the Geometric R.V.

Recall: $X \sim Geo(p)$

If we toss the coin once every minute,
then X measures how many minutes we'll
need to wait until we get heads.

Let n be a large integer,
Suppose instead we flip a coin every $\frac{1}{n}$ minute
but with success probability $\frac{p}{n}$.

Exponential Distribution

Let $\lambda > 0$ be a parameter

For $n \in \mathbb{N}$ we toss a coin every $\frac{1}{n}$ minutes

w. heads prob. $\frac{\lambda}{n}$.

Y – the time you wait until you get heads.

$$\Pr\left[Y = \frac{i}{n}\right] = \frac{\lambda}{n} \cdot \left(1 - \frac{\lambda}{n}\right)^{i-1}$$

$$\Pr\left[Y > \frac{i}{n}\right] = \left(1 - \frac{\lambda}{n}\right)^{i}$$

For $y > 0$ $\Pr\left[Y > y\right] = \Pr\left[Y > \frac{y \cdot n}{n}\right] = \left(1 - \frac{\lambda}{n}\right)^{yn} \xrightarrow[n \to \infty]{} e^{-\lambda y}.$

Definition – the Exponential Distribution

A r.v. $Y \sim Exp(\lambda)$ if for all $y > 0$

$$\Pr[Y > y] = e^{-\lambda y}.$$

In other words, $F_Y(y) = \begin{cases} 0, & \text{if } y \le 0 \\ 1 - e^{-\lambda y}, & \text{if } y > 0 \end{cases}$

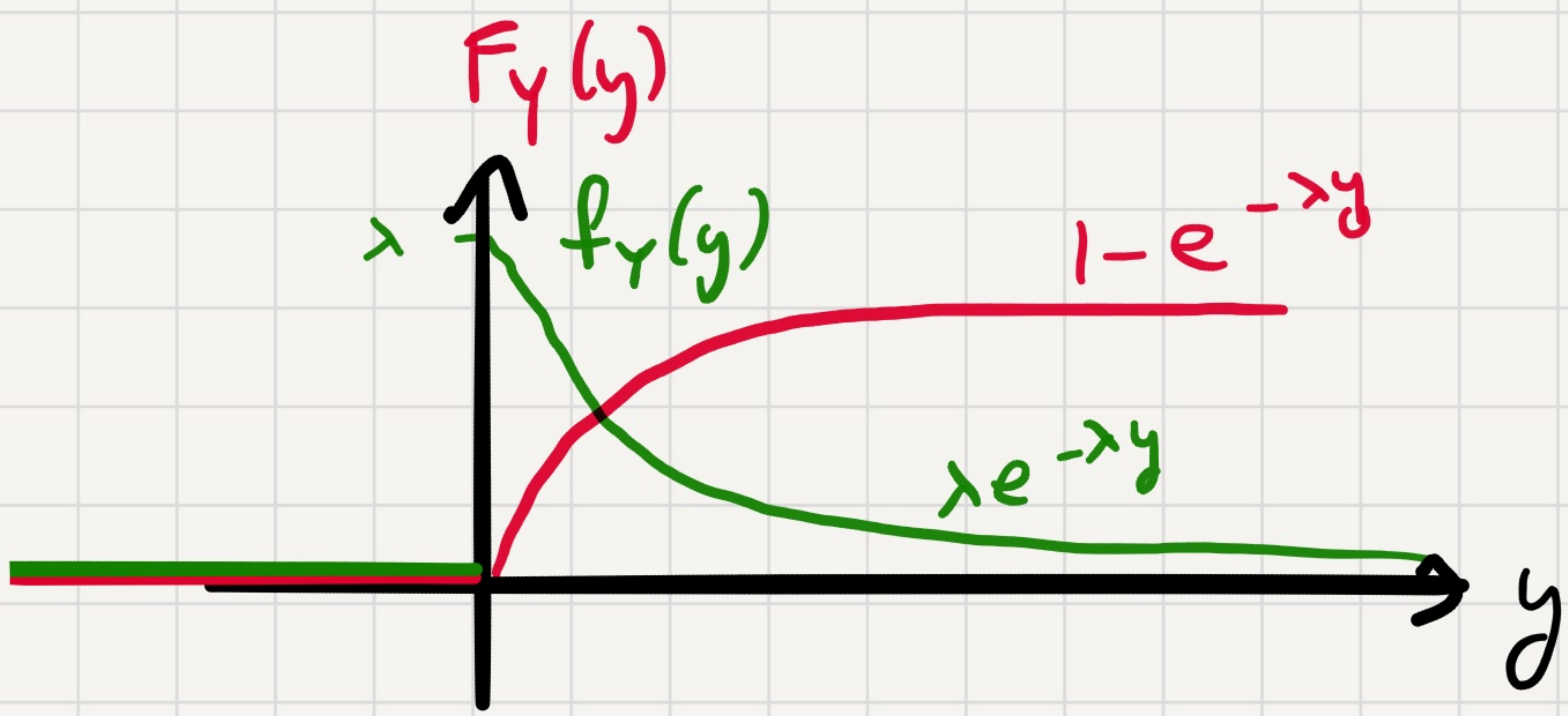
and $f_Y(y) = \begin{cases} 0, & \text{if } y \le 0 \\ \lambda \cdot e^{-\lambda y}, & \text{if } y > 0 \end{cases}$

Definition – the Exponential Distribution

A r.v. $Y \sim Exp(\lambda)$

has $F_Y(y) = \begin{cases} 0, & \text{if } y \leq 0 \\ 1-e^{-\lambda y}, & \text{if } y > 0 \end{cases}$

and $f_Y(y) = \begin{cases} 0, & \text{if } y \leq 0 \\ \lambda \cdot e^{-\lambda y}, & \text{if } y > 0 \end{cases}$



Definition – the Exponential Distribution

A r.v. $Y \sim Exp(\lambda)$ if for all $y > 0$

$$Pr[Y > y] = e^{-\lambda y}.$$

$Exp(\lambda)$ is memoryless

For $s, t \geq 0$

$$Pr[Y > s+t \mid Y > s] = \frac{Pr[Y > s+t]}{Pr[Y > s]}$$

$$= \frac{e^{-\lambda(s+t)}}{e^{-\lambda s}} = e^{-\lambda t} = Pr[Y > t]$$

Definition - the Exponential Distribution

A r.v. $Y \sim Exp(\lambda)$ if for all $y > 0$

$$\Pr[Y > y] = e^{-\lambda y}.$$

Exp(λ) is memoryless

Suppose the lifetime of a mouse is distributed $X \sim Exp(1)$

$$\Pr[X > 5 \mid X > 4] =$$

$$\Pr[X > 20 \mid X > 19] =$$

Expectation of a Continuous Random Variable

If X is a r.v. with p.d.f $f_X(x)$, then the **expectation** of X is

$$\mathbb{E}[X] = \int_{-\infty}^{\infty} f_X(x) \cdot x \cdot dx.$$

Justification: Use discrete approximation $Y = \frac{\lceil X \cdot n \rceil}{n}$

$$\mathbb{E}[Y] = \sum_{i=-\infty}^{\infty} \frac{i}{n} \cdot \Pr\left[\frac{i-1}{n} < X \leq \frac{i}{n}\right] \xrightarrow[n \to \infty]{} \int_{-\infty}^{\infty} x \cdot f(x) dx$$

Variance of a Continuous Random Variable

If X is a r.v. with p.d.f $f(x)$, then the Variance of X is

$$\mathrm{Var}(X) = \mathbb{E}[X^2] - \mathbb{E}(X)^2$$

$$= \int_{-\infty}^{\infty} x^2 \cdot f(x)\,dx - \left(\int_{-\infty}^{\infty} x \cdot f(x)\,dx\right)^2$$

Expectation of Exp(λ)

$X \sim \mathrm{Exp}(\lambda)$ $\qquad f(x) = \begin{cases} \lambda \cdot e^{-\lambda x}, & 0 \le x \\ 0, & \text{o.w.} \end{cases}$

$$\mathbb{E}[X] = \int_0^\infty x \cdot \lambda e^{-\lambda x}\, dx = x \cdot \left(-e^{-\lambda x}\right)\Big|_0^\infty - \int_0^\infty \left(-e^{-\lambda x}\right) dx$$

$$= 0 - \frac{e^{-\lambda x}}{\lambda}\Big|_0^\infty = \frac{1}{\lambda}.$$

Integration by Parts:

$$\int_a^b u(x) \cdot v'(x)\, dx = u(x) \cdot v(x)\Big|_a^b - \int_a^b u'(x) \cdot v(x)\, dx$$

Expectation and Variance of $Exp(\lambda)$

$X \sim Exp(\lambda)$ $\qquad$ $f(x) = \begin{cases} \lambda \cdot e^{-\lambda x}, & 0 \leq x \\ 0, & o.w. \end{cases}$

$$\mathbb{E}[X] = \int_0^\infty x \cdot \lambda e^{-\lambda x} dx = x \cdot (-e^{-\lambda x}) \Big|_0^\infty - \int_0^\infty (-e^{-\lambda x}) dx$$

$$= 0 - \frac{e^{-\lambda x}}{\lambda} \Big|_0^\infty = \frac{1}{\lambda}.$$

$$\mathbb{E}[X^2] = \int_0^\infty x^2 \cdot \lambda e^{-\lambda x} = x^2 \cdot (-e^{-\lambda x}) \Big|_0^\infty + \int_0^\infty 2x \cdot e^{-\lambda x} dx$$

$$= 0 + \frac{2}{\lambda} \int_0^\infty \lambda x e^{-\lambda x} dx$$

$$= \frac{2}{\lambda} \cdot \mathbb{E}X = \frac{2}{\lambda^2}$$

Expectation and Variance of Exp(λ)

$X \sim \text{Exp}(\lambda)$

$$f(x) = \begin{cases} \lambda \cdot e^{-\lambda x}, & 0 \le x \\ 0, & \text{o.w.} \end{cases}$$

$\mathbb{E}[X] = \frac{1}{\lambda}$

$\mathbb{E}[X^2] = \frac{2}{\lambda^2}$

$\text{Var}(X) = \frac{2}{\lambda^2} - \frac{1}{\lambda^2} = \frac{1}{\lambda^2}.$

This shouldn't surprise us:

Recall: $X = \lim_{n \to \infty} \frac{Z_n}{n}$

$Z_n \sim \text{Geo}\left(\frac{\lambda}{n}\right)$

$$\lim_{n \to \infty} \mathbb{E}\left[\frac{Z_n}{n}\right] = \lim_{n \to \infty} \frac{n/\lambda}{n} = \frac{1}{\lambda}$$

$$\lim_{n \to \infty} \text{Var}\left[\frac{Z_n}{n}\right] = \lim_{n \to \infty} \frac{\text{Var}(Z_n)}{n^2}$$

$$= \lim_{n \to \infty} \frac{1}{n^2} \cdot \frac{1 - \frac{\lambda}{n}}{(\lambda/n)^2} = \frac{1}{\lambda^2}.$$

Expectation of Uniform Distribution

$$X \sim U([0,\ell])$$

$$f_X(x) = \begin{cases} 1/\ell, & 0 \leq x \leq \ell \\ 0, & \text{o.w.} \end{cases}$$

$$\mathbb{E}[X] = \int_0^\ell x \cdot \frac{1}{\ell}\, dx = \left.\frac{x^2}{2\ell}\right|_0^\ell = \frac{\ell^2}{2\ell} = \frac{\ell}{2}$$

$$\mathbb{E}[X^2] = \int_0^\ell X^2 \cdot \frac{1}{\ell}\, dx = \left.\frac{x^3}{3\ell}\right|_0^\ell = \frac{\ell^3}{3\ell} = \frac{\ell^2}{3}$$

$$\mathrm{Var}(X) = \mathbb{E}(X^2) - \mathbb{E}[X]^2 = \frac{\ell^2}{3} - \frac{\ell^2}{4} = \frac{\ell^2}{12}.$$

Joint Density

If X and Y are continuous r.v.s then their joint p.d.f is a non-negative function $f_{X,Y}(x,y)$

s.t. $$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dx\,dy$$

and for all $a<b$, $c<d$

$$\Pr[a \leq X \leq b, c \leq Y \leq d] = \int_c^d \int_a^b f_{X,Y}(x,y)\,dx\,dy$$

Note:

$$f_{X,Y}(x,y) = \lim_{\substack{dx\to 0\\ dy \to 0}} \frac{\Pr[x \leq X \leq x+dx,\ y \leq Y \leq y+dy]}{dx \cdot dy}$$

Independent Random Variables

X and Y are **indep.** continuous r.v.s if

for all $a < b$, $c < d$

$$\Pr[a \le X \le b,\ c \le Y \le d] = \Pr[a \le X \le b] \cdot \Pr[c \le Y \le d]$$

Equivalently, if $f_{X,Y}(x, y) = f_X(x) \cdot f_Y(y)$.

for all x, y.