

# Lecture 23

## Continuous Probability I

## Plan for Today

1. Motivating Example - uniform  $X \in [0,1]$ .
2. Probability of events
3. Continuous Random Variables & Distributions.
4. Definitions: cumulative distribution function  
probability density function.
5. Exponential RVs.

Uniformly at random in  $[0,1]$ .

Choose a real number  $X$ , uniformly at random in  $[0,1]$ .

Q: What's the probability that  $X = 0.3$ ?

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What about  $\Pr[X \leq \frac{1}{2}] = ?$

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What about  $\Pr[x \leq \frac{1}{2}] = ?$

$\Pr[x \in [0.3, 0.4]] = ?$

$\Pr[x \in [a, b]] = ?$

## Uniformly at random in $[0,1]$ .

Choose a real number  $X$ , uniformly at random in  $[0,1]$ .

For any  $0 \leq a < b \leq 1$  the event " $X \in [a,b]$ " has prob.  $b-a$ .

$$\Pr [X \in [0,0.1] \text{ or } X \in [0.8,0.9]] = ?$$

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$$\begin{aligned} & \Pr [X \in [0, 0.1] \text{ or } X \in [0.8, 0.9]] \\ &= \Pr [X \in [0, 0.1]] + \Pr [X \in [0.8, 0.9]] = 0.2. \end{aligned}$$

In General: If  $E_1, E_2, E_3, \dots$  are disjoint events

$$\Pr [E_1 \cup E_2 \cup \dots] = \sum_i \Pr [E_i]$$



## Uniformly at random in $[0,1]$ .

Choose a real number  $x$ , uniformly at random in  $[0,1]$ .

Define

$$F_x(x) = \Pr[X \leq x]$$

$$F_x(x) = \begin{cases} 0, & \text{if } x < 0 \\ x, & \text{if } 0 \leq x \leq 1 \\ 1, & \text{if } 1 < x \end{cases}$$

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$$\begin{aligned} \Pr[X \in (a,b)] &= \Pr[X \leq b] - \Pr[X \leq a] \\ &= F_x(b) - F_x(a). \end{aligned}$$

$\Rightarrow F_x$  specifies Prob. of all events

## Definition:

Let  $X$  be a real valued random variable.

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## Notes:

1. A r.v. is continuous if  $F_X$  is continuous.

2. The c.d.f. defines the prob. of each interval

$$\Pr[X \in (a, b]] = F_X(b) - F_X(a)$$

3. Hence, the c.d.f. defines the prob. of each event (union of intervals).

# Discrete vs Continuous Probability

## Discrete

Finite or Countable  
Sample Space  $\Omega$

Start with  $\Pr[\omega]$

Derive prob. of events:

$$\Pr[E] = \sum_{\omega \in E} \Pr[\omega]$$

## Continuous

Continuous Sample Space.

$\Pr[\omega]$  is typically 0.

Start with prob. of events,

e.g. intervals,

union of intervals.

## Uniformly at random in $[0,1]$ - Discrete Approximation

$X$  uniformly random in  $[0,1]$

Let  $n$  be a large integer.

$$Y = \frac{\lceil X \cdot n \rceil}{n}.$$

In other words, if  $\frac{i-1}{n} < X \leq \frac{i}{n}$   
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$$\Pr[Y = \frac{i}{n}] = \Pr[X \in (\frac{i-1}{n}, \frac{i}{n}]] = \frac{1}{n}$$

$Y$  is uniform in  $\{\frac{1}{n}, \frac{2}{n}, \frac{3}{n}, \dots, 1\}$

$$|X - Y| \leq \frac{1}{n}$$

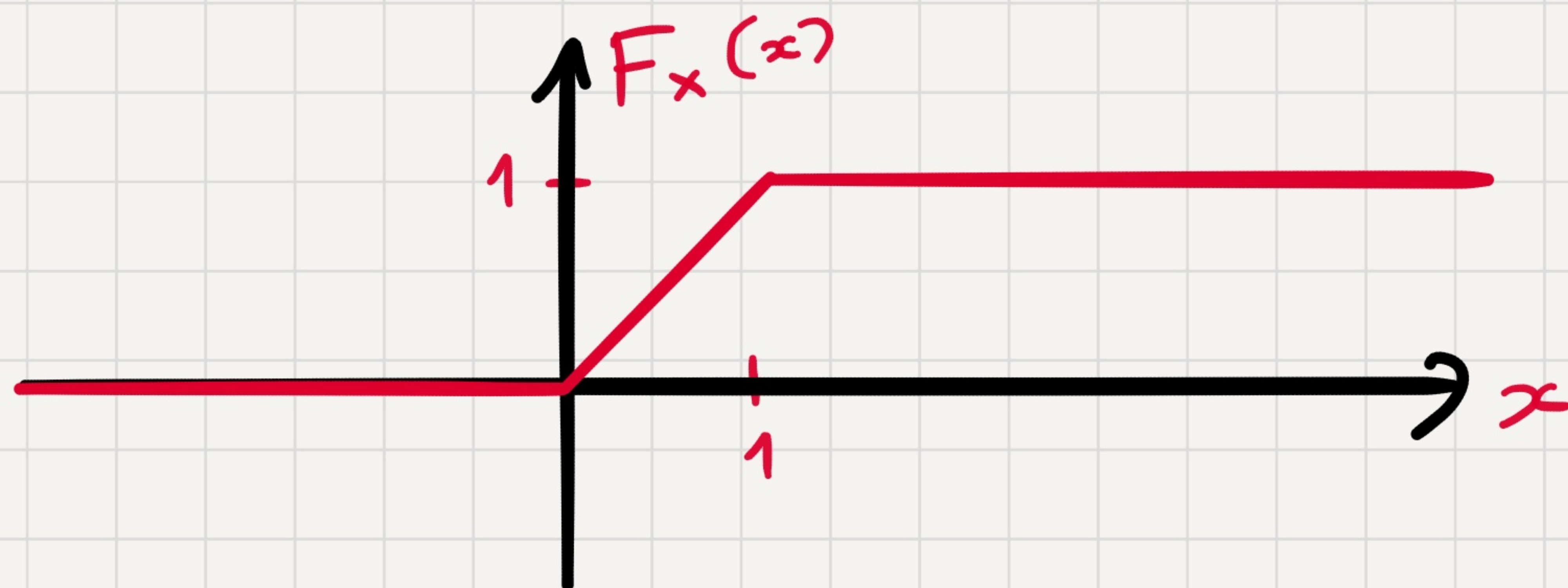
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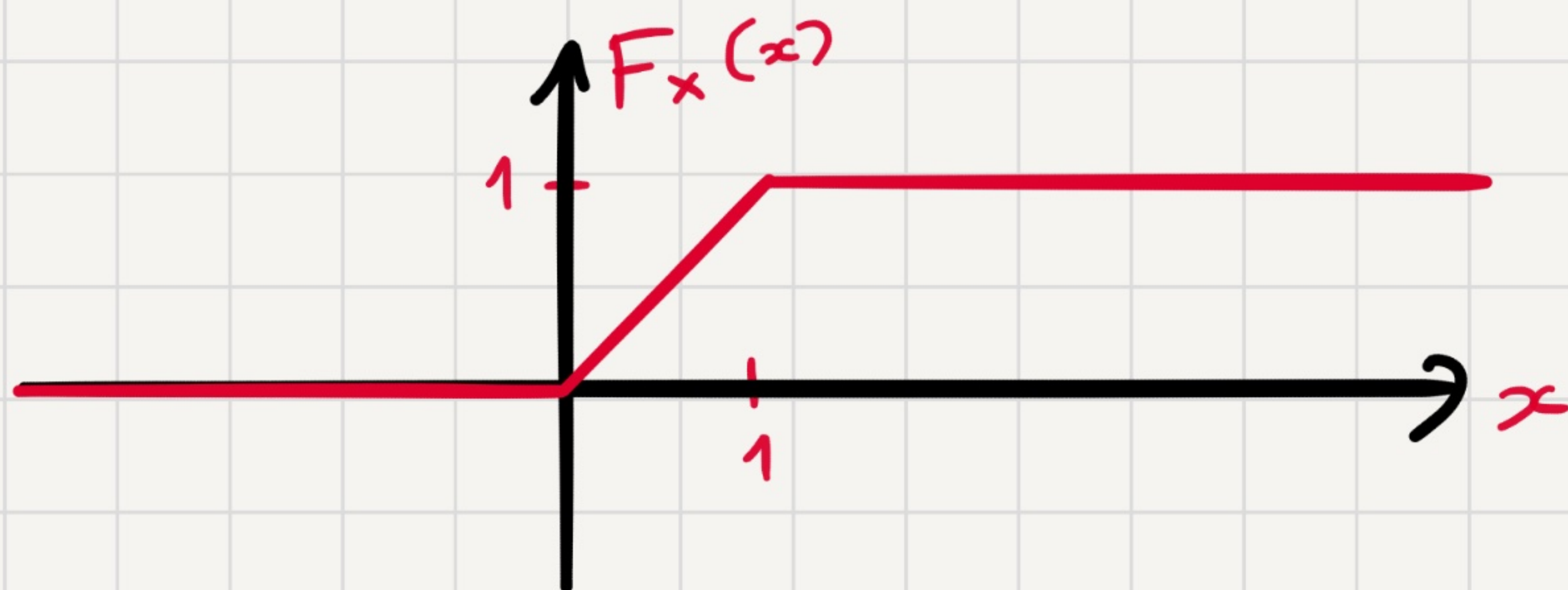




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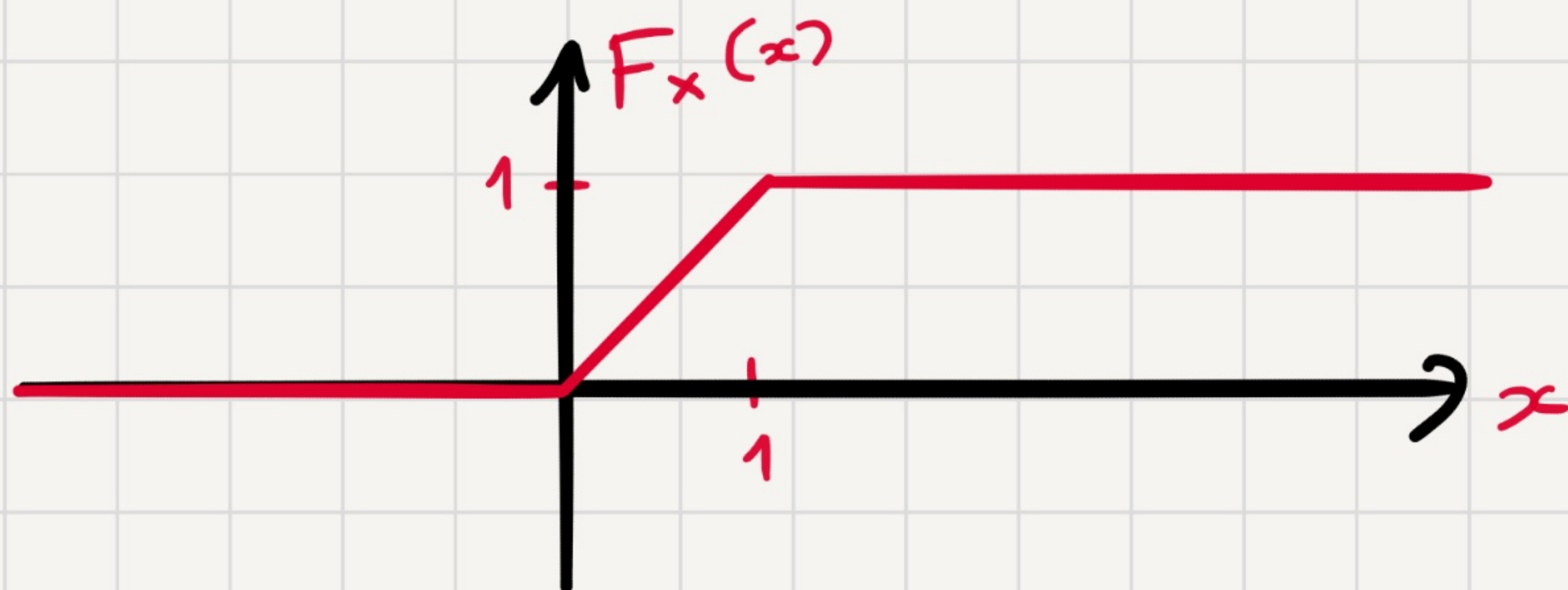
Define  $f_X(x) = \lim_{\delta \rightarrow 0^+} \frac{\Pr[X \in (x, x+\delta)]}{\delta}$

probability  
density  
function

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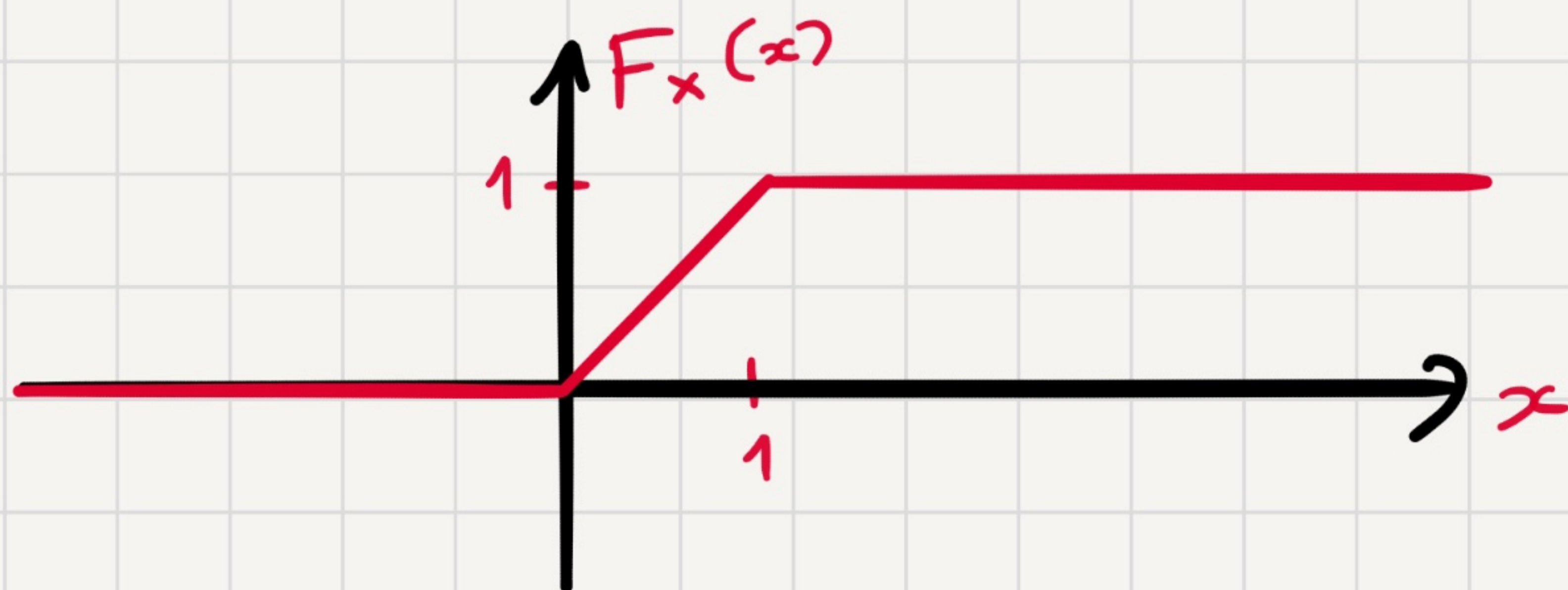
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Then  $f_X(x) = \lim_{\delta \rightarrow 0^+} \frac{F_X(x+\delta) - F_X(x)}{\delta} = ?$

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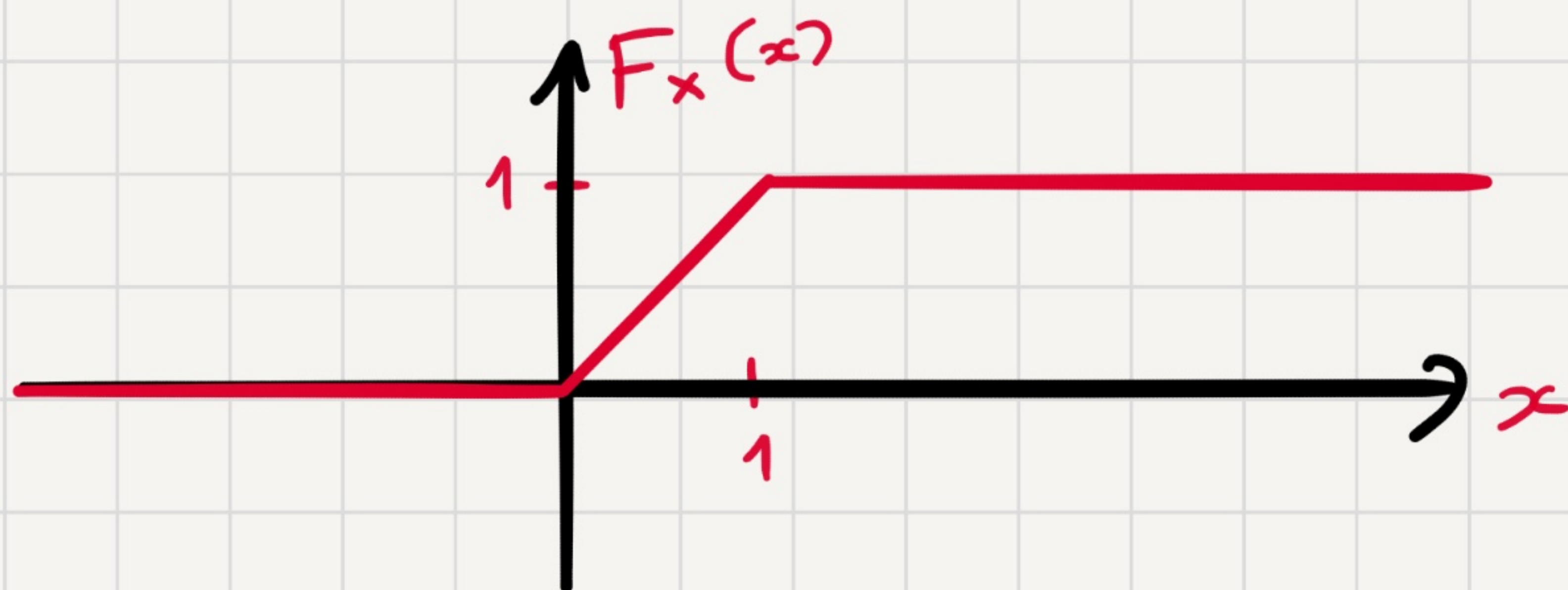
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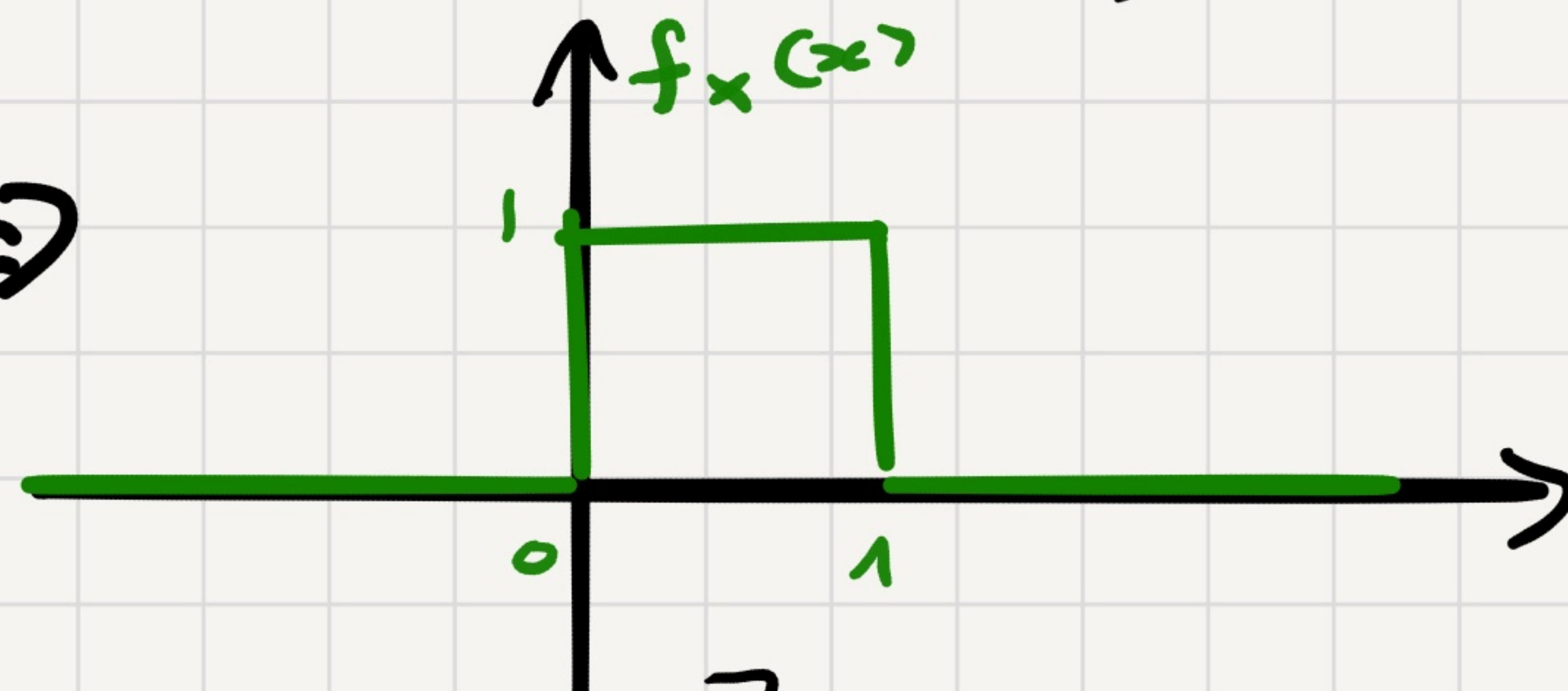
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$\Rightarrow$



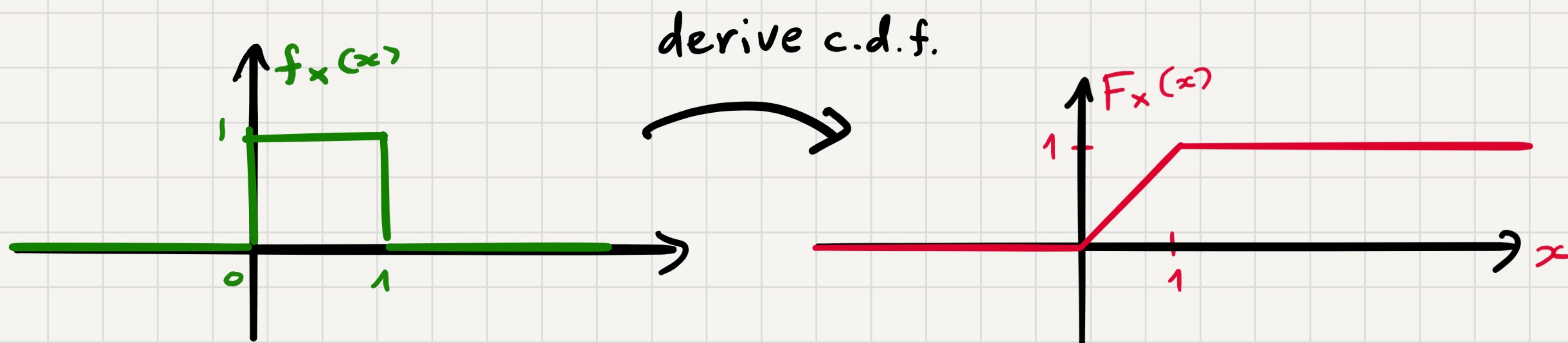
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then,  $f_X(x) = F'_X(x) = \begin{cases} 0 & , \text{ if } x < 0 \\ 1 & , \text{ if } 0 \leq x < 1 \\ 0 & , \text{ if } 1 \leq x \end{cases}$

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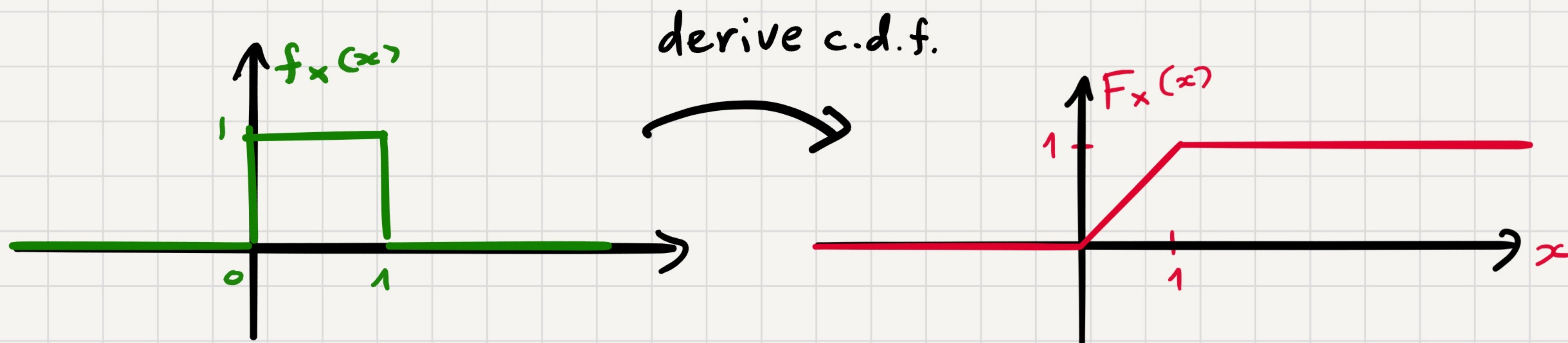
Alternative Derivation: Start with the density function and derive the c.d.f.



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$$\text{If } f_X(x) = F_X'(x)$$

$$\text{Then, } F_X(b) - F_X(a) = \int_a^b f_X(x) dx.$$

$$F_X(a) = \int_{-\infty}^a f_X(x) dx$$

## Definition:

Let  $X$  be a real valued random variable

The cumulative distribution function (c.d.f.)  
of  $X$  is

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$$f_X(x) = F'_X(x)$$

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For small  $dx$ ,

$$\Pr[X \in [x, x+dx)] \approx f(x) dx$$

# Properties of Density Functions

1.  $f_X(x)$  is non-negative

2.  $\int_{-\infty}^{\infty} f_X(x) dx = \boxed{\phantom{000}}$

Q: Can  $f_X(x)$  be larger than 1?

## Alternative Definition

Let  $X$  be a real valued random variable.

The probability density function (p.d.f) of  $X$

is a non-negative function  $f(x)$  s.t.  $\int_{-\infty}^{\infty} f(x) dx = 1$

and for all  $a < b$ :

$$\Pr[a \leq X \leq b] = \int_a^b f(x) dx.$$

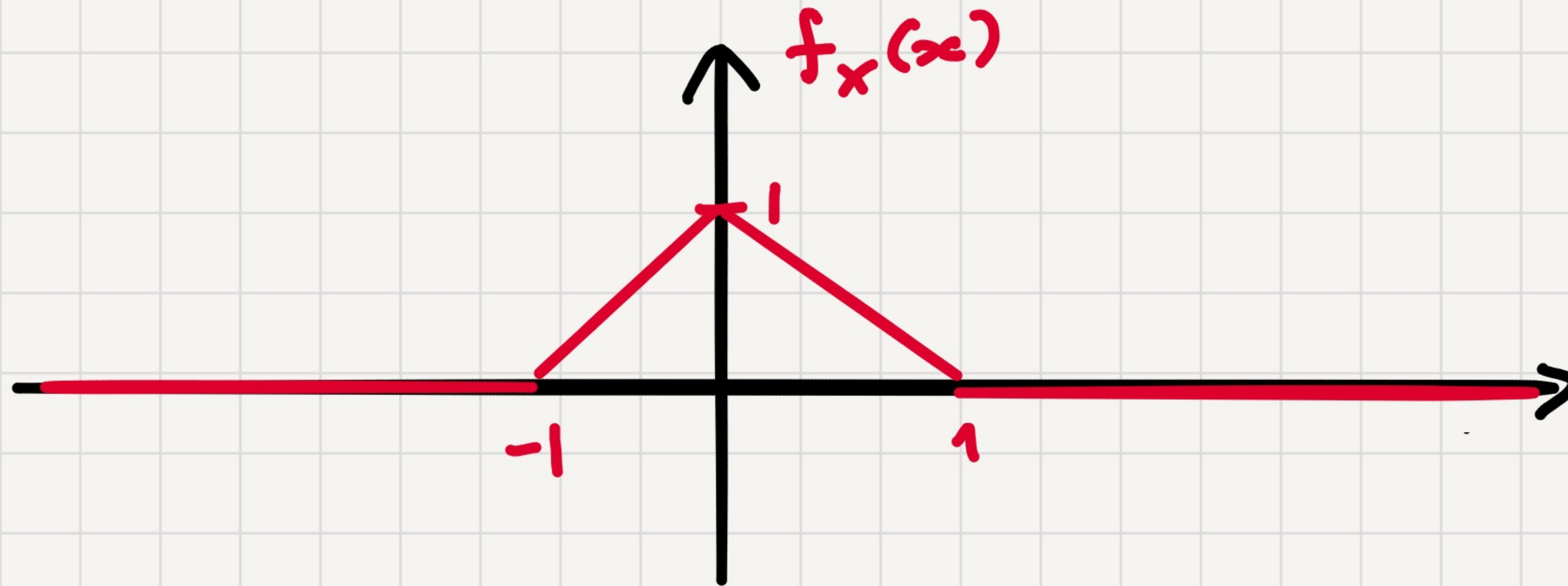
## Uniform in $[0, l]$

$$F_x(x) = \boxed{\phantom{0.5x}} \quad \text{for } 0 \leq x \leq l$$

$$f_x(x) = \begin{cases} \boxed{\phantom{0.5}}, & \text{for } 0 \leq x \leq l \\ \boxed{\phantom{0}}, & \text{o.w.} \end{cases}$$

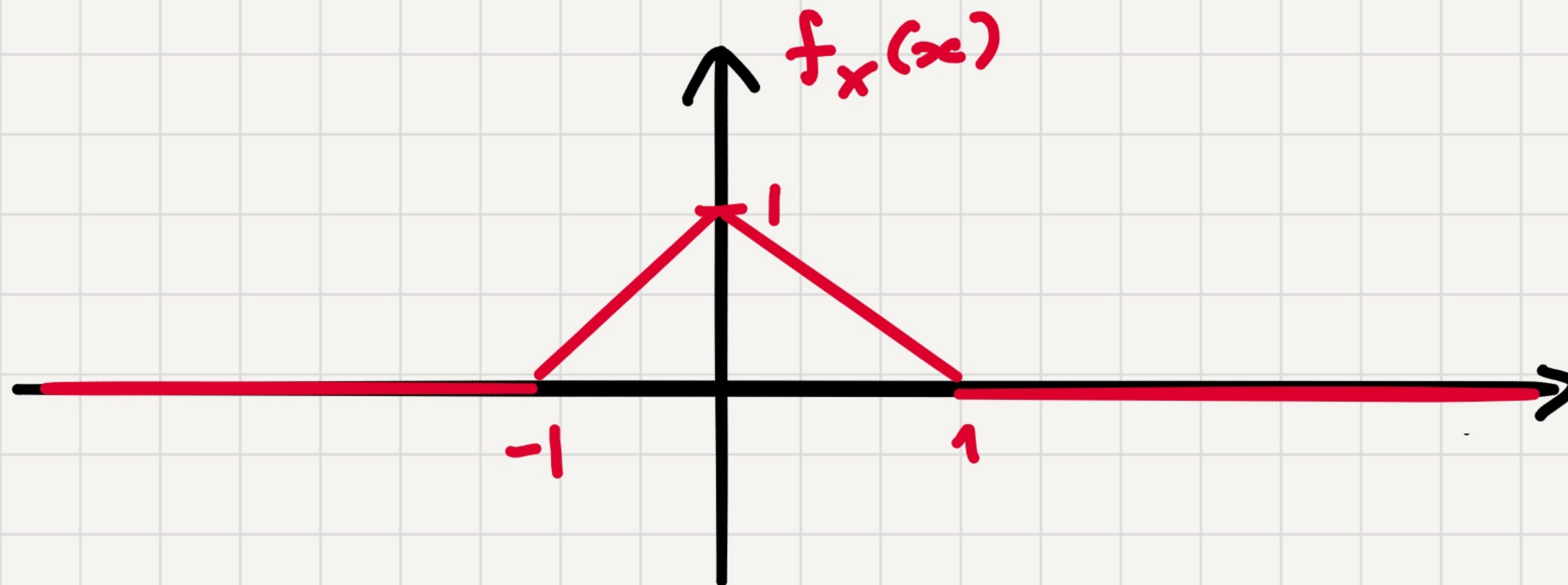
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$$f_X(x) = \begin{cases} 0 & \text{if } x < -1 \\ x+1 & \text{if } -1 \leq x < 0 \\ 1-x & \text{if } 0 \leq x < 1 \\ 0 & \text{if } 1 \leq x \end{cases}$$

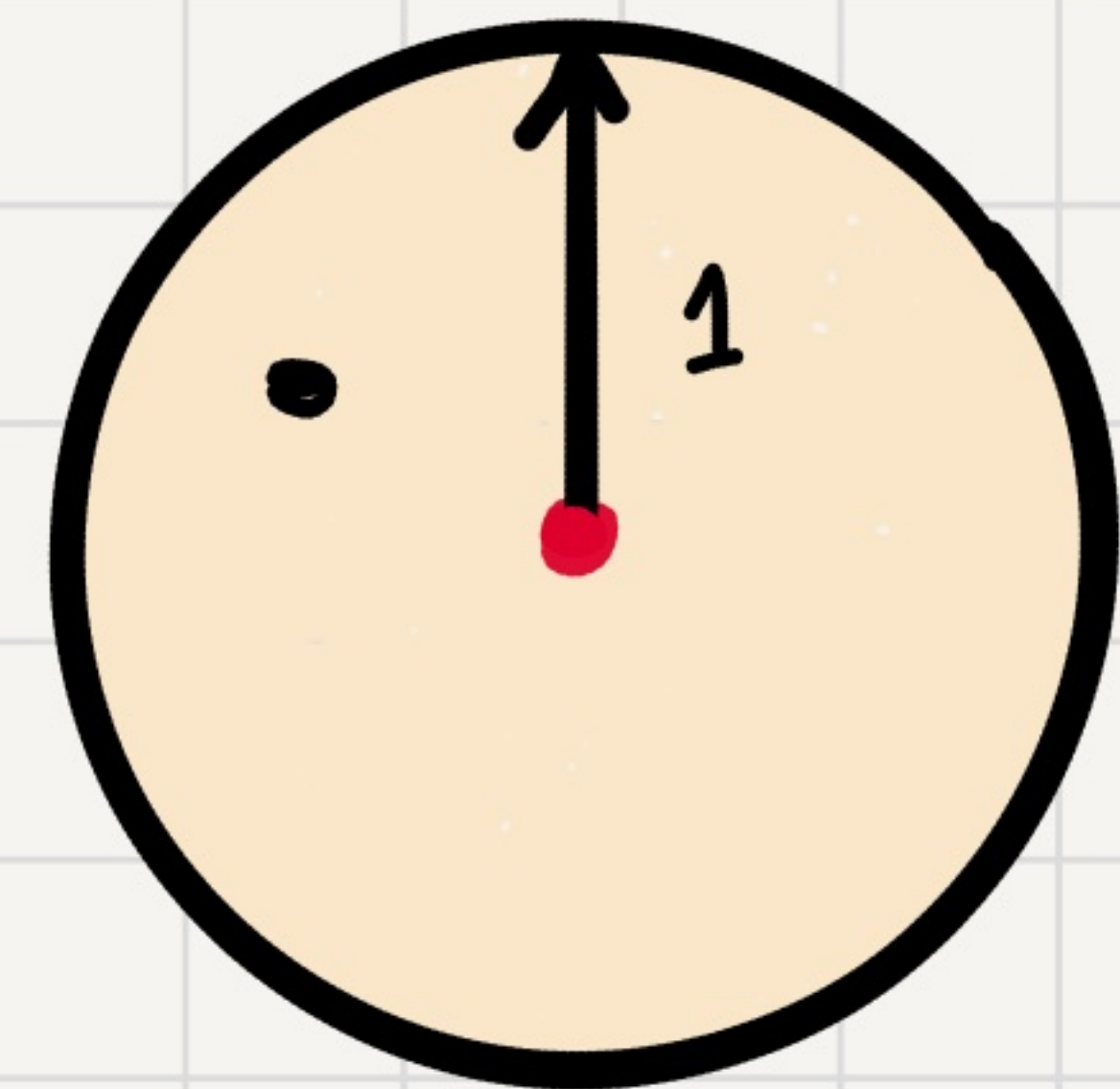
$$\int_{-\infty}^{\infty} f_X(x) dx = \text{area under curve} = 1.$$

## Example #3

You throw a dart at a dartboard at random and you hit it.  $X$  - distance from center.

What's the distribution of  $X$ ?

$$\Pr[X \leq x] =$$

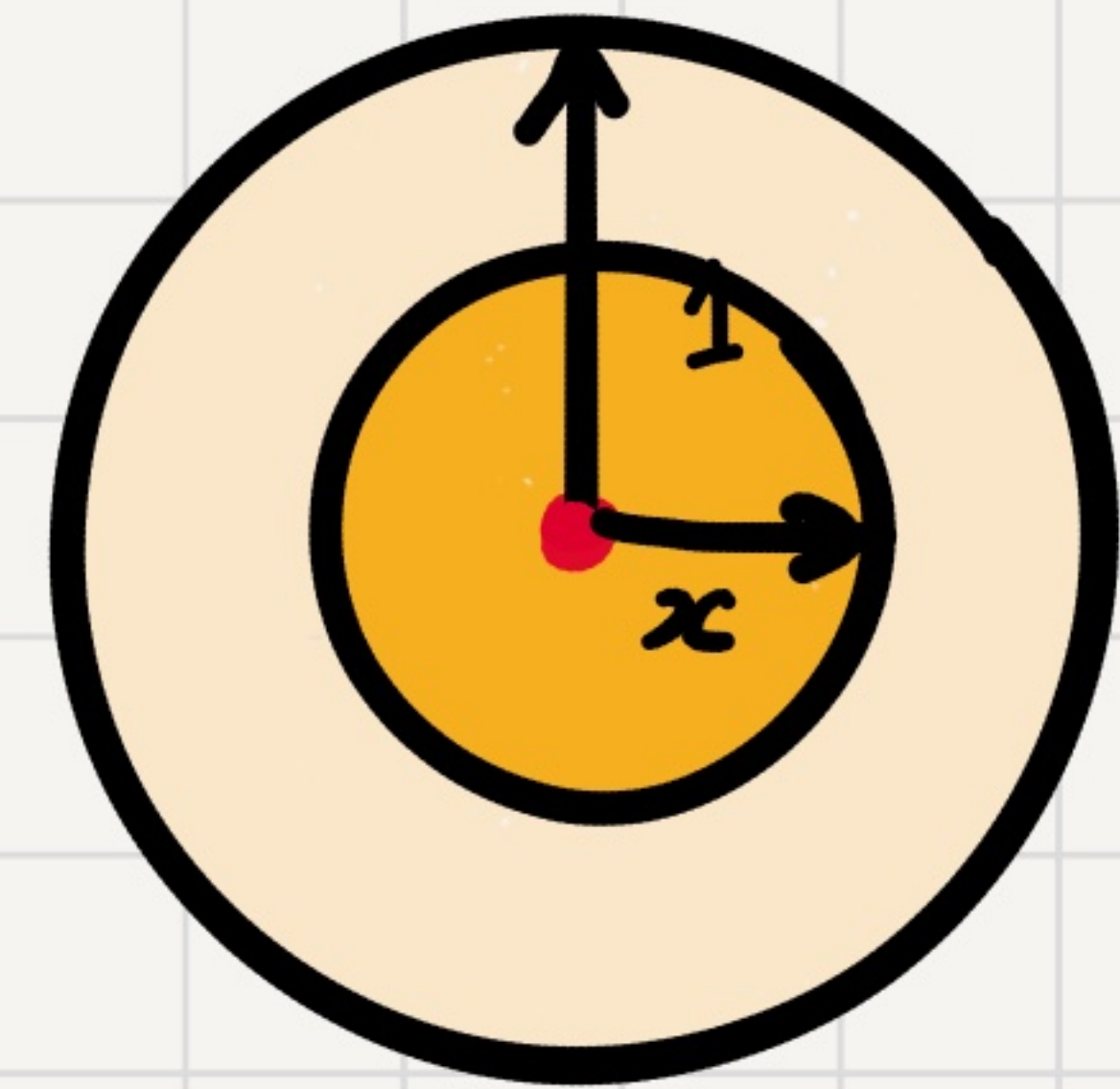


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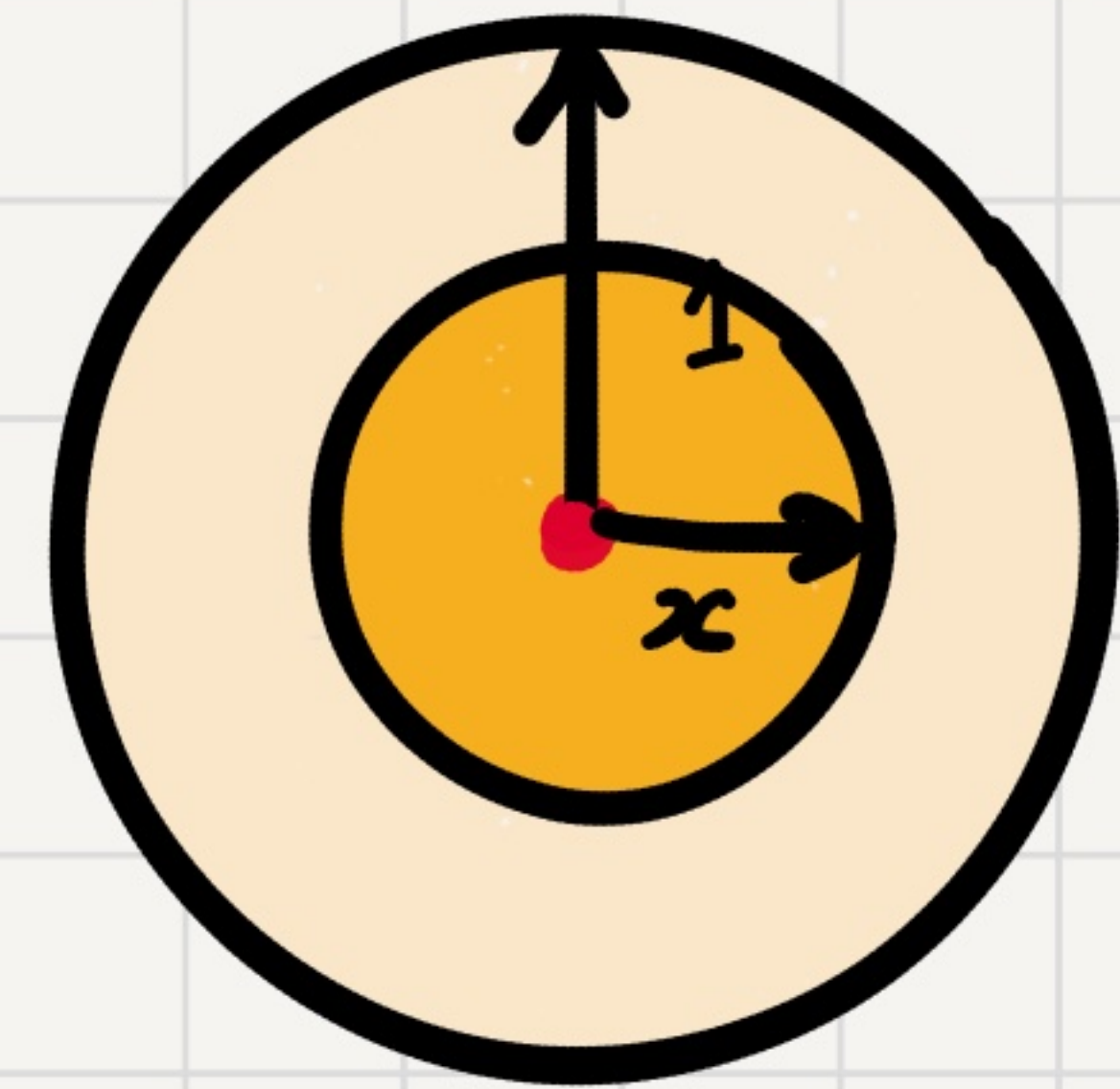
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$$= \frac{\pi x^2}{\pi \cdot 1^2} = x^2$$



$$F_X(x) = \begin{cases} 0, & x < 0 \\ x^2, & 0 \leq x < 1 \\ 1, & 1 \leq x \end{cases}$$

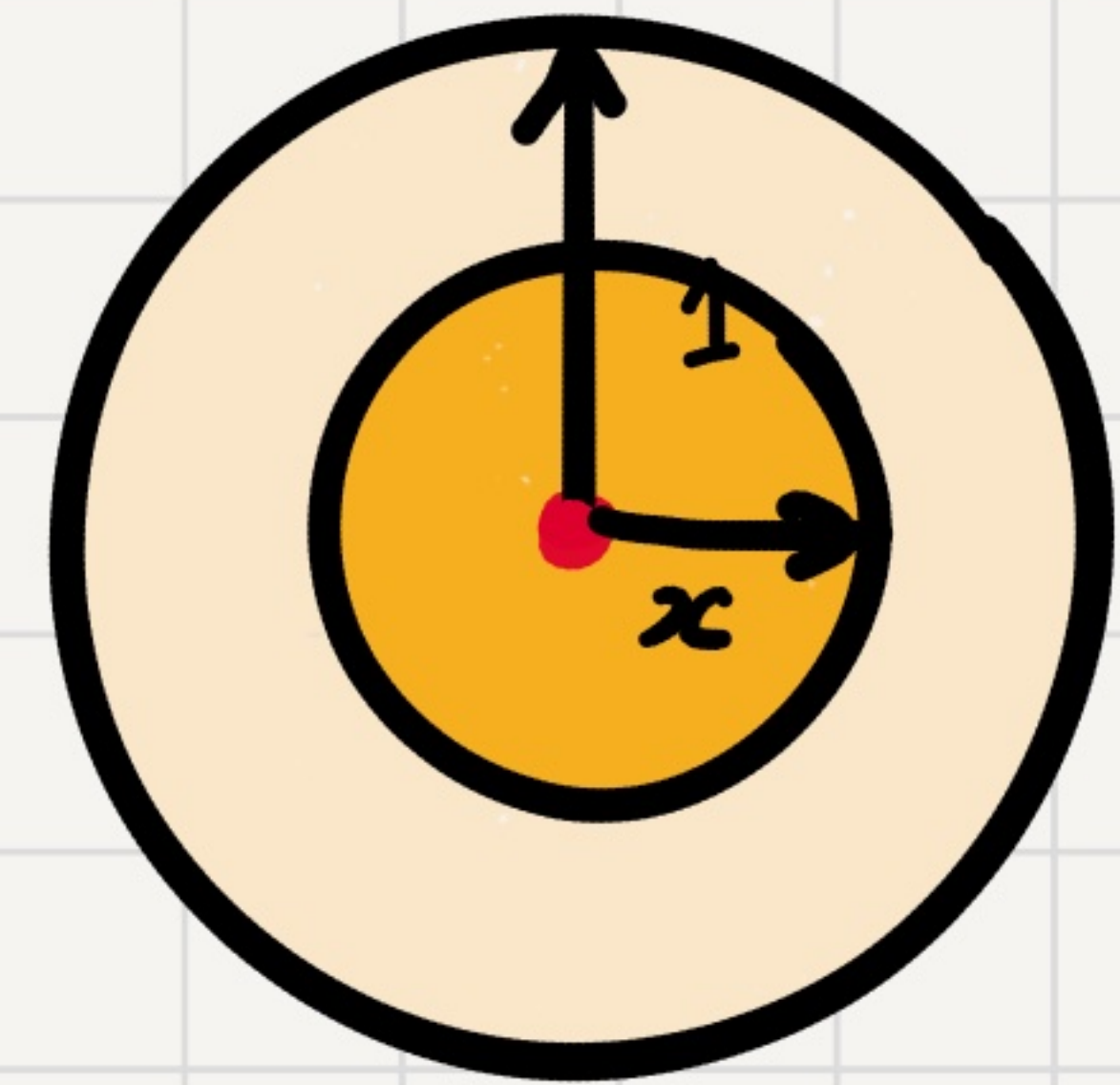
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$$f_X(x) = ?$$

$$F_X(x) = \begin{cases} 0, & x < 0 \\ x^2, & 0 \leq x < 1 \\ 1, & 1 \leq x \end{cases}$$

# Exponential Random Variable

This is the analog to the Geometric R.V.

Recall:  $X \sim \text{Geo}(p)$

measures how many trials until we get H

where  $\Pr[H] = p$ .

$$\Pr[X=i] = (1-p)^{i-1} \cdot p.$$

$$\Pr[X > i] = (1-p)^i$$

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If we toss the coin once every minute,  
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If we toss the coin once every minute,  
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need to wait until we get heads.

Let  $n$  be a large integer,

suppose instead we flip a coin every  $\frac{1}{n}$  minute  
but with success probability  $\frac{p}{n}$ .

# Exponential Distribution

Let  $\lambda > 0$  be a parameter

For  $n \in \mathbb{N}$ , we toss a coin every  $\frac{1}{n}$  minutes  
w. heads prob.  $\frac{\lambda}{n}$ .

$Y$  - the time you wait until you get heads.

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$$\Pr \left[ Y = \frac{i}{n} \right] = \frac{\lambda}{n} \cdot \left( 1 - \frac{\lambda}{n} \right)^{i-1}$$

$$\Pr \left[ Y > \frac{i}{n} \right] = \left( 1 - \frac{\lambda}{n} \right)^i$$

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$$\Pr\left[Y > \frac{i}{n}\right] = \left(1 - \frac{\lambda}{n}\right)^i$$

For  $y > 0$   $\Pr[Y > y] = \Pr\left[Y > \frac{y \cdot n}{n}\right] = \left(1 - \frac{\lambda}{n}\right)^{y \cdot n}$   
 $\xrightarrow{n \rightarrow \infty} e^{-\lambda y}$ .



## Definition - the Exponential Distribution

A r.v.  $Y \sim \text{Exp}(\lambda)$  if for all  $y > 0$

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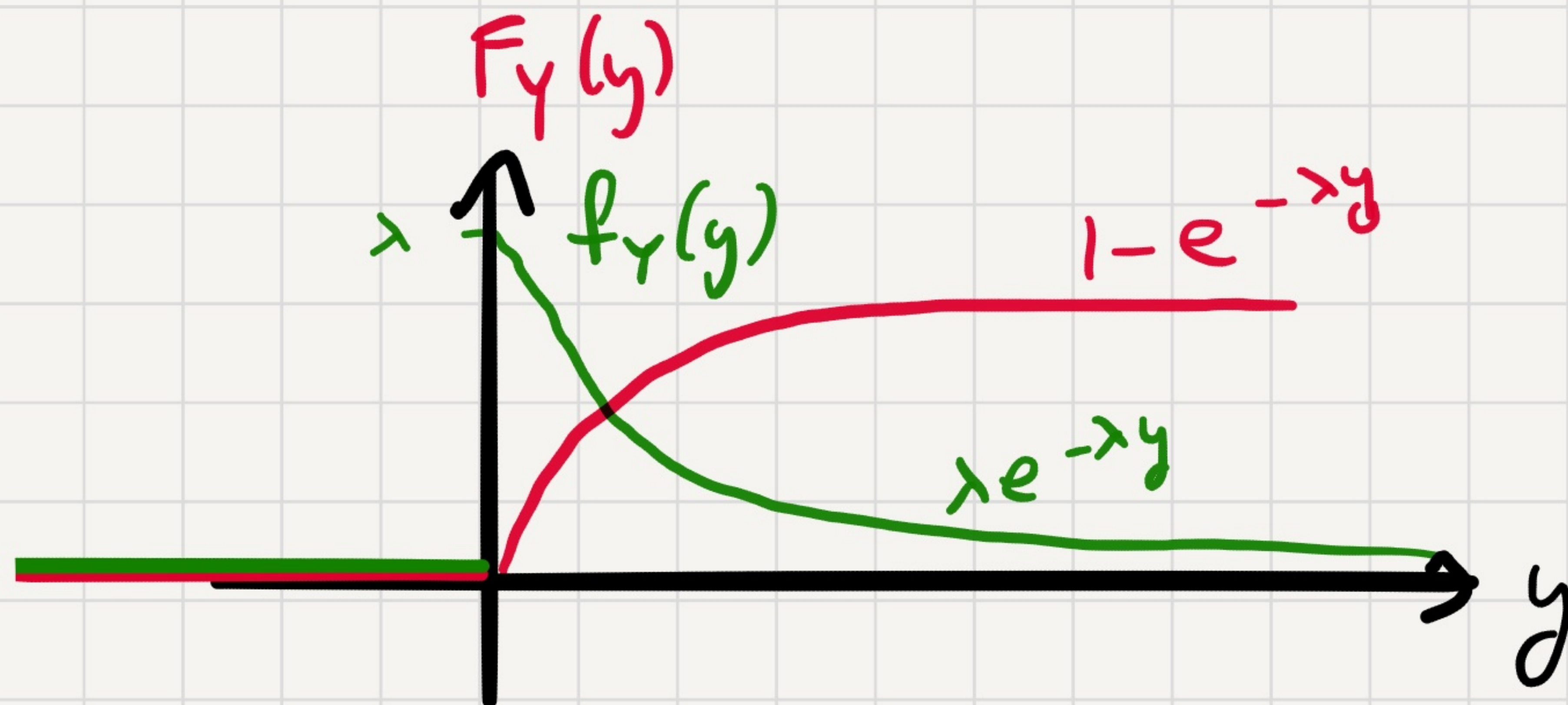
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Exp( $\lambda$ ) is memoryless

For  $s, t \geq 0$

$$\begin{aligned} \Pr[Y > s+t \mid Y > s] &= \frac{\Pr[Y > s+t]}{\Pr[Y > s]} \\ &= \frac{e^{-\lambda(s+t)}}{e^{-\lambda s}} = e^{-\lambda t} = \Pr[Y > t] \end{aligned}$$

## Definition - the Exponential Distribution

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$\text{Exp}(\lambda)$  is memoryless

Suppose the lifetime of a mouse is distributed  $x \sim \text{Exp}(1)$

$$\Pr[X > 5 \mid X > 4] =$$

$$\Pr[X > 20 \mid X > 19] =$$

## Expectation of a Continuous Random Variable

If  $X$  is a r.v. with p.d.f  $f_X(x)$ , then

the expectation of  $X$  is

$$E[X] = \int_{-\infty}^{\infty} f_X(x) \cdot x \cdot dx.$$

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Justification:

Use discrete approximation

$$Y = \frac{\lceil X \cdot n \rceil}{n}$$

$$E[Y] = \sum_{i=-\infty}^{\infty} \frac{i}{n} \cdot \Pr \left[ \frac{i}{n} < X \leq \frac{i+1}{n} \right] \xrightarrow{n \rightarrow \infty}$$

$$\int_{-\infty}^{\infty} x \cdot f(x) dx$$



## Variance of a Continuous Random Variable

If  $X$  is a r.v. with p.d.f  $f(x)$ , then  
the Variance of  $X$  is

$$\text{Var}(X) = \mathbb{E}[X^2] - (\mathbb{E}[X])^2$$

$$= \int_{-\infty}^{\infty} x^2 \cdot f(x) dx - \left( \int_{-\infty}^{\infty} x \cdot f(x) dx \right)^2$$

## Expectation of $\text{Exp}(\lambda)$

$$X \sim \text{Exp}(\lambda)$$

$$f(x) = \begin{cases} \lambda \cdot e^{-\lambda x} & , 0 \leq x \\ 0 & , \text{o.w.} \end{cases}$$

$$E[X] = ?$$

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$$X \sim \text{Exp}(\lambda)$$

$$f(x) = \begin{cases} \lambda \cdot e^{-\lambda x} & , 0 \leq x \\ 0 & , o.w. \end{cases}$$

$$E[X] = \int_0^{\infty} x \cdot \lambda e^{-\lambda x} dx = ?$$

### Integration by Parts:

$$\int_a^b u(x) \cdot v'(x) dx = u(x) \cdot v(x) \Big|_a^b - \int_a^b u'(x) \cdot v(x) dx$$

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$$f(x) = \begin{cases} \lambda \cdot e^{-\lambda x} & , 0 \leq x \\ 0 & , o.w. \end{cases}$$

$$\begin{aligned} \mathbb{E}[X] &= \int_0^{\infty} x \cdot \lambda e^{-\lambda x} dx = x \cdot (-e^{-\lambda x}) \Big|_0^{\infty} - \int_0^{\infty} (-e^{-\lambda x}) dx \\ &= 0 - \frac{e^{-\lambda x}}{\lambda} \Big|_0^{\infty} = \frac{1}{\lambda} \end{aligned}$$

### Integration by Parts:

$$\int_a^b u(x) \cdot v'(x) dx = u(x) \cdot v(x) \Big|_a^b - \int_a^b u'(x) \cdot v(x) dx$$

## Expectation and Variance of $\text{Exp}(\lambda)$

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$$\mathbb{E}[X^2] = \int_0^{\infty} x^2 \cdot \lambda e^{-\lambda x} = ?$$

## Expectation and Variance of $\text{Exp}(\lambda)$

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$$\mathbb{E}[X^2] = \int_0^{\infty} x^2 \cdot \lambda e^{-\lambda x} = ?$$

## Expectation and Variance of $\text{Exp}(\lambda)$

$$X \sim \text{Exp}(\lambda)$$

$$f(x) = \begin{cases} \lambda \cdot e^{-\lambda x}, & 0 \leq x \\ 0, & \text{o.w.} \end{cases}$$

$$\begin{aligned} \mathbb{E}[X] &= \int_0^{\infty} x \cdot \lambda e^{-\lambda x} dx = x \cdot (-e^{-\lambda x}) \Big|_0^{\infty} - \int_0^{\infty} (-e^{-\lambda x}) dx \\ &= 0 - \frac{e^{-\lambda x}}{\lambda} \Big|_0^{\infty} = \frac{1}{\lambda}. \end{aligned}$$

$$\begin{aligned} \mathbb{E}[X^2] &= \int_0^{\infty} x^2 \cdot \lambda e^{-\lambda x} dx = x^2 \cdot (-e^{-\lambda x}) \Big|_0^{\infty} + \int_0^{\infty} 2x \cdot e^{-\lambda x} dx \\ &= 0 + \frac{2}{\lambda} \int_0^{\infty} \lambda x e^{-\lambda x} dx \\ &= \frac{2}{\lambda} \cdot \mathbb{E}X = \frac{2}{\lambda^2} \end{aligned}$$

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$$\mathbb{E}[X] = \frac{1}{\lambda}$$

$$\mathbb{E}[X^2] = \frac{2}{\lambda^2}$$

$$\text{Var}(X) = ?$$



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$$E[X] = \frac{1}{\lambda}$$

$$E[X^2] = \frac{2}{\lambda^2}$$

$$\text{Var}(X) = \frac{2}{\lambda^2} - \left(\frac{1}{\lambda}\right)^2 = \frac{1}{\lambda^2}$$

This shouldn't surprise us:

Recall:  $X = \lim_{n \rightarrow \infty} \frac{Z_n}{n}$

$$Z_n \sim \text{Geo}\left(\frac{\lambda}{n}\right)$$

$$\lim_{n \rightarrow \infty} E\left[\frac{Z_n}{n}\right] = \lim_{n \rightarrow \infty} \frac{n/\lambda}{n} = \frac{1}{\lambda}$$

$$\lim_{n \rightarrow \infty} \text{Var}\left(\frac{Z_n}{n}\right) = \lim_{n \rightarrow \infty} \frac{\text{Var}(Z_n)}{n^2}$$

$$= \lim_{n \rightarrow \infty} \frac{1}{n^2} \cdot \frac{1 - \frac{\lambda}{n}}{\left(\frac{\lambda}{n}\right)^2} = \frac{1}{\lambda^2}$$

## Expectation of Uniform Distribution

$$X \sim U([0, l])$$

$$f_X(x) = \begin{cases} 1/l, & 0 \leq x \leq l \\ 0, & \text{o.w.} \end{cases}$$

$$E[X] =$$

$$E[X^2] =$$

$$\text{Var}(X) =$$

## Expectation of Uniform Distribution

$$X \sim U([0, l])$$

$$f_X(x) = \begin{cases} 1/l, & 0 \leq x \leq l \\ 0, & \text{o.w.} \end{cases}$$

$$E[X] = \int_0^l x \cdot \frac{1}{l} dx = \left. \frac{x^2}{2l} \right|_0^l = \frac{l^2}{2l} = \frac{l}{2}$$

$$E[X^2] = \int_0^l x^2 \cdot \frac{1}{l} dx = \left. \frac{x^3}{3l} \right|_0^l = \frac{l^3}{3l} = \frac{l^2}{3}$$

$$\text{Var}(X) = E[X^2] - (E[X])^2 = \frac{l^2}{3} - \frac{l^2}{4} = \frac{l^2}{12}$$

# Joint Density

If  $X$  and  $Y$  are continuous r.v.s then their

joint p.d.f is a non-negative function  $f_{X,Y}(x,y)$

s.t. 
$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{X,Y}(x,y) dx dy$$

and for all  $a < b, c < d$

$$\Pr[a \leq X \leq b, c \leq Y \leq d] = \int_c^d \int_a^b f_{X,Y}(x,y) dx dy$$

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Note:

$$f_{X,Y}(x,y) = \lim_{\substack{dx \rightarrow 0 \\ dy \rightarrow 0}}$$

$$\frac{\Pr[x \leq X \leq x+dx, y \leq Y \leq y+dy]}{dx \cdot dy}$$

## Independent Random Variables

$X$  and  $Y$  are indep. continuous r.v.s if

for all  $a < b$ ,  $c < d$

$$\Pr[a \leq X \leq b, c \leq Y \leq d] = \Pr[a \leq X \leq b] \cdot \Pr[c \leq Y \leq d]$$

Equivalently, if  $f_{X,Y}(x,y) = f_X(x) \cdot f_Y(y)$ .

for all  $x, y$ .