

Lecture 24

Continuous Probability II

Recap

Continuous Probability

Continuous (uncountable) sample space. Continuous R.V.s.

$\forall x: \Pr[X=x]=0$. Instead, we define Prob. of intervals.

c.d.f. $F_X(x) = \Pr[X \leq x]$

$$\Pr[a \leq X \leq b] = F_X(b) - F_X(a)$$

p.d.f. $f_X(x) = \lim_{dx \rightarrow 0} \frac{F_X(x+dx) - F_X(x)}{dx} = F'_X(x)$

$$\Pr[a \leq X \leq b] = \int_a^b f_X(x) dx.$$

• Uniform dist. in $[0,1]$

$$f_X(x) = \begin{cases} 1, & 0 \leq x \leq 1 \\ 0, & \text{o.w.} \end{cases}$$

• Exponential dist (λ)

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x}, & x \geq 0 \\ 0, & \text{o.w.} \end{cases}$$

$$\Pr[X > x] = e^{-\lambda x}$$

Expectation of a Continuous Random Variable

If X is a r.v. with p.d.f $f_X(x)$, then

the expectation of X is

$$E[X] = \int_{-\infty}^{\infty} f_X(x) \cdot x \cdot dx.$$

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Justification: Use discrete approximation $Y = \frac{\lceil X \cdot n \rceil}{n}$

$$E[Y] = \sum_{i=-\infty}^{\infty} \frac{i}{n} \cdot \Pr\left[\frac{i}{n} < X \leq \frac{i+1}{n}\right] \xrightarrow{n \rightarrow \infty} \int_{-\infty}^{\infty} x \cdot f(x) dx$$

Expectation of a Continuous Random Variable

If X is a r.v. with p.d.f $f_X(x)$, then
the expectation of X is

$$E[X] = \int_{-\infty}^{\infty} f_X(x) \cdot x \cdot dx.$$

For any function $g: \mathbb{R} \rightarrow \mathbb{R}$

$$E[g(X)] = \int_{-\infty}^{\infty} g(x) \cdot f_X(x) dx$$

Joint Distributions

If X and Y are continuous r.v.s then their

joint p.d.f is a non-negative function $f(x, y)$

s.t.
$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) dx dy$$

and for all $a < b, c < d$

$$\Pr[a \leq X \leq b, c \leq Y \leq d] = \int_c^d \int_a^b f(x, y) dx dy$$

Marginals:

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x, y) dy$$

$$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x, y) dx.$$

Joint Distributions

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$$\Pr[a \leq X \leq b, c \leq Y \leq d] = \int_c^d \int_a^b f(x, y) dx dy$$

Note:

$$f(x, y) = \lim_{\substack{dx \rightarrow 0 \\ dy \rightarrow 0}} \frac{\Pr[x \leq X \leq x+dx, y \leq Y \leq y+dy]}{dx \cdot dy}$$

Independent Random Variables

X and Y are indep. continuous r.v.s if

for all $a < b$, $c < d$

$$\Pr[a \leq X \leq b, c \leq Y \leq d] = \Pr[a \leq X \leq b] \cdot \Pr[c \leq Y \leq d]$$

Equivalently, if $f_{XY}(x, y) = f_X(x) \cdot f_Y(y)$.

for all x, y .

Linearity of Expectation

If x_1, \dots, x_n continuous r.v.s then

$$E[x_1 + \dots + x_n] = E[x_1] + \dots + E[x_n].$$

Independent Random Variables

1. If X and Y are indep. then $E[XY] = E[X] \cdot E[Y]$.

2. If X and Y are indep. then

$$\text{Var}(X+Y) = \text{Var}(X) + \text{Var}(Y).$$

Example

$$X \sim \text{Exp}(\lambda) \quad Y \sim \text{Exp}(\mu) \quad X, Y \text{ indep.}$$

What's $\Pr[X > Y] = ?$

Example

$$X \sim \text{Exp}(\lambda) \quad Y \sim \text{Exp}(\mu) \quad X, Y \text{ indep.}$$

What's $\Pr[X > Y] = ?$

$$\Pr[X > Y] = \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} f_{X,Y}(x,y) dx \right] dy$$

$$= \int_{-\infty}^{\infty} \left[\int_0^{\infty} f_X(x) f_Y(y) dx \right] dy$$

$$= \int_{-\infty}^{\infty} \mu \cdot e^{-\mu y} \left[\int_0^{\infty} x e^{-\lambda x} dx \right] dy$$

$$= \int_{-\infty}^{\infty} \mu e^{-\mu y} \cdot e^{-\lambda y} dy = \frac{\mu}{\mu + \lambda} \int_{-\infty}^{\infty} (\mu + \lambda) e^{-(\mu + \lambda)y} dy$$

$$= \frac{\mu}{\mu + \lambda}.$$

Example - Meeting at a restaurant

Alice & Bob try to meet for lunch between 12 and 1

Each arrives at a uniformly random time within the hour
and will wait 10 minutes.

Q: what's the probability they meet?

Example - Meeting at a restaurant

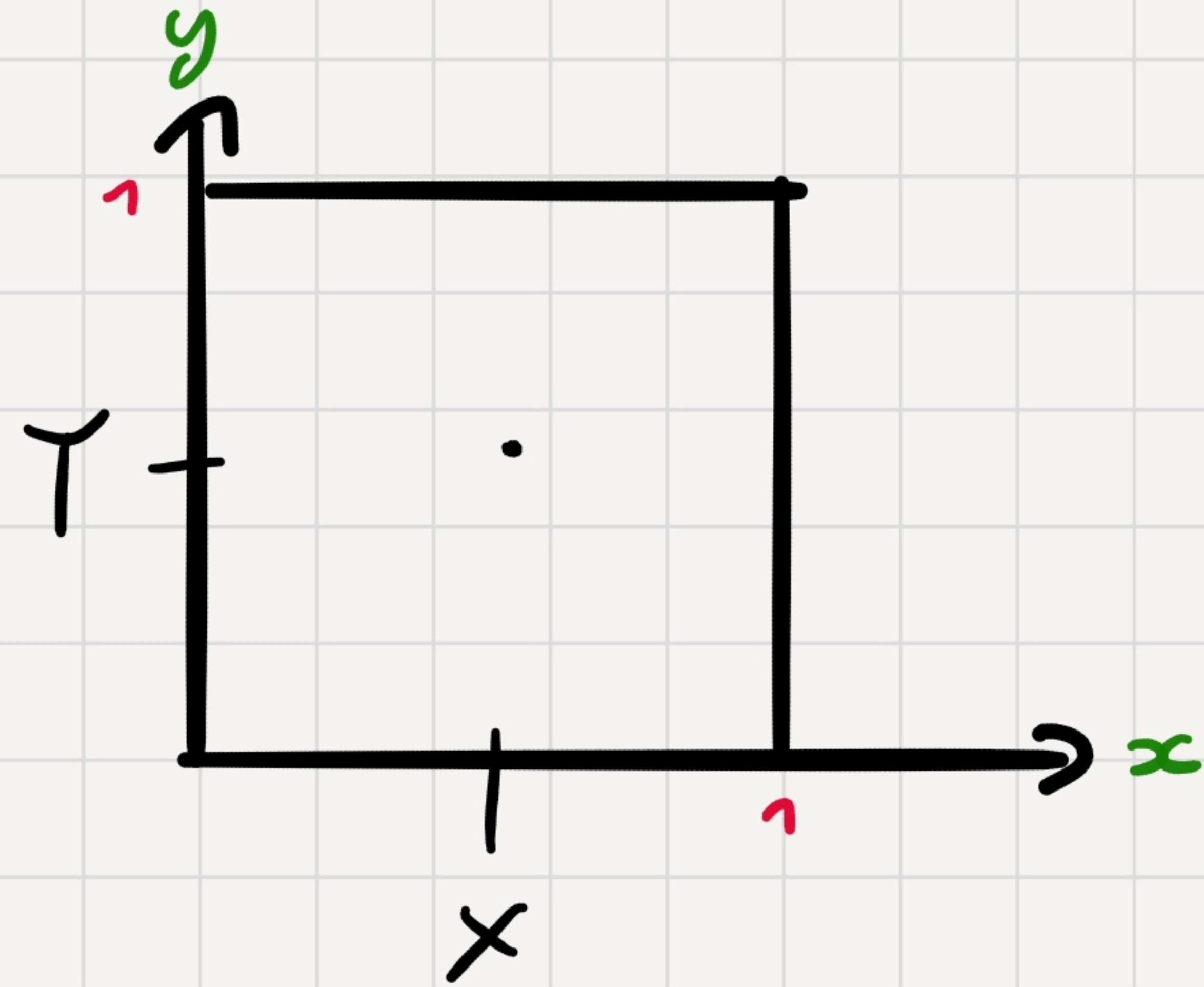
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Q: what's the probability they meet?

X, Y indep. and $U([0,1])$

$$f_{X,Y}(x,y) = \begin{cases} 1, & \text{if } 0 \leq x, y \leq 1 \\ 0, & \text{o.w.} \end{cases}$$



$$\Pr[\text{meet}] =$$

Example - Meeting at a restaurant

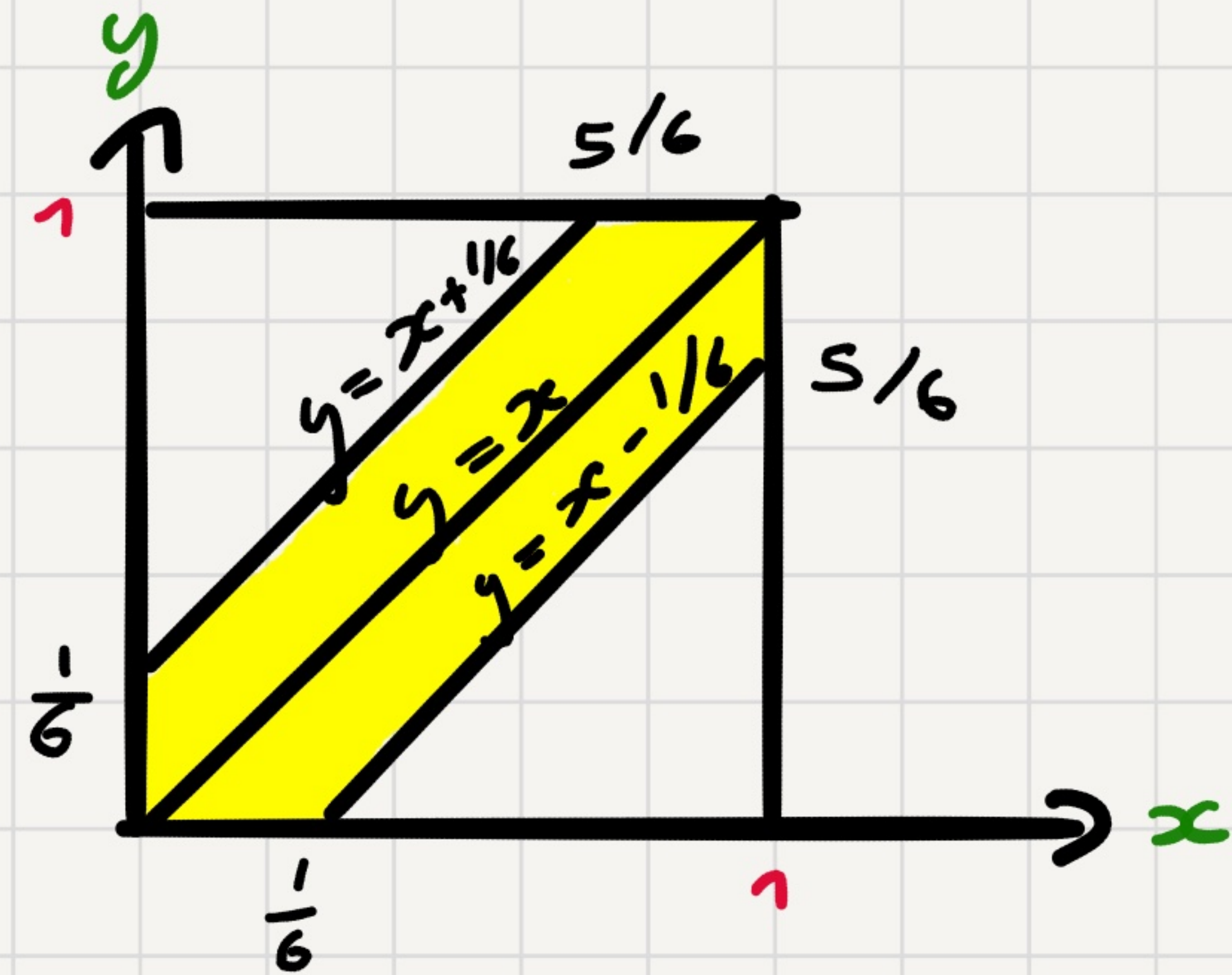
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$$\Pr[\text{meet}] = \frac{\text{area of shaded region}}{\text{area of square}}$$

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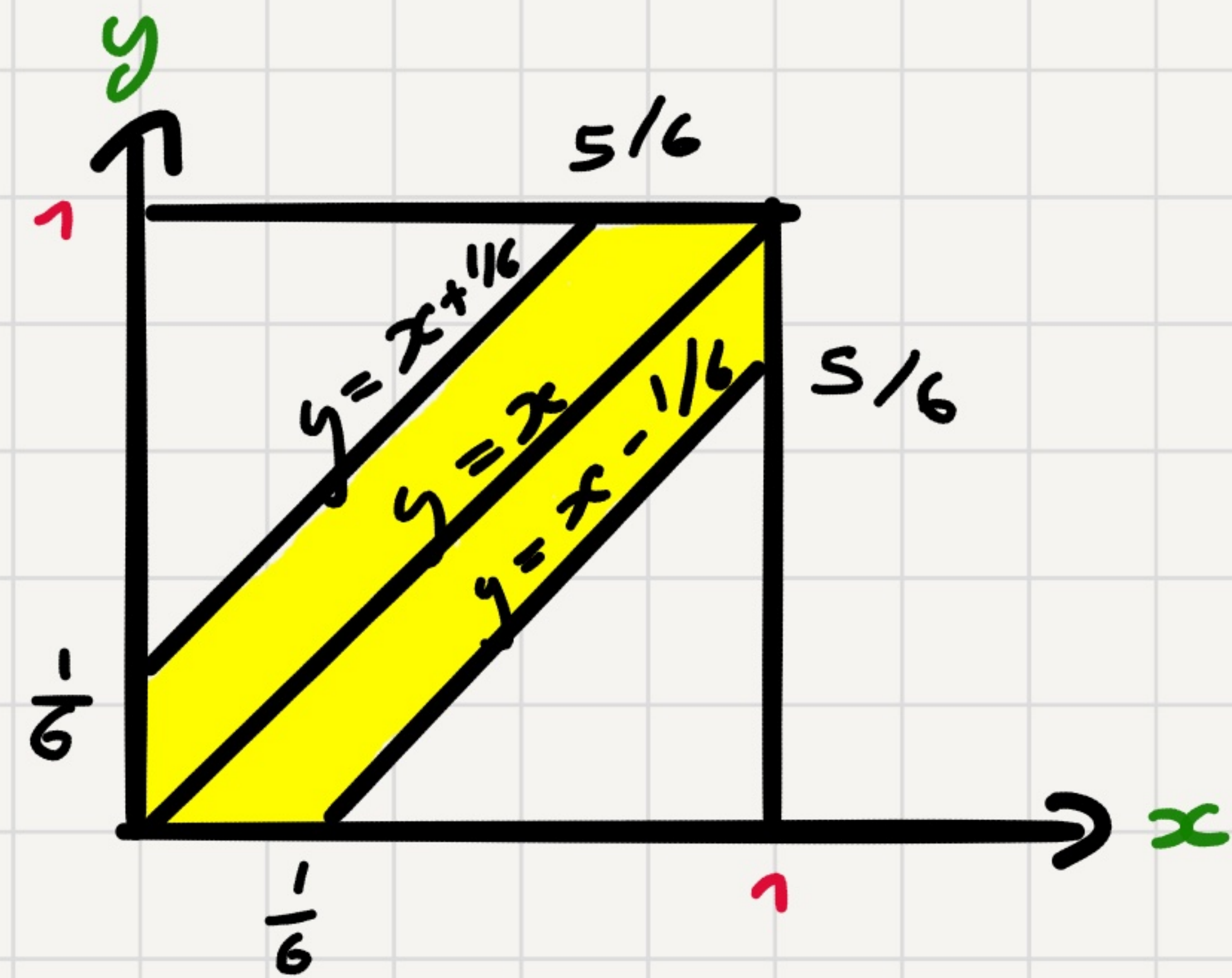
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$$f_{X,Y}(x,y) = \begin{cases} 1, & \text{if } 0 \leq x, y \leq 1 \\ 0, & \text{o.w.} \end{cases}$$



$$\Pr[\text{meet}] = \frac{\text{area of shaded region}}{\text{area of square}} = 1 - \frac{(5/6)^2}{2} - \frac{(5/6)^2}{2} = \frac{11}{36}.$$

Normal (Gaussian) Distributions

A r.v. X is distributed $N(\mu, \sigma^2)$ if its p.d.f is

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

Normal (Gaussian) Distributions

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A r.v. X is a standard Gaussian if $X \sim N(0, 1)$

and its p.d.f. is

$$f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}.$$

Normal (Gaussian) Distributions

A r.v. X is distributed $N(\mu, \sigma^2)$ if its p.d.f is

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

Theorem: If $X \sim N(0, 1)$ then $Y = \sigma X + \mu$
is dist. $N(\mu, \sigma^2)$.

If $Y \sim N(\mu, \sigma^2)$ then $X = \frac{Y - \mu}{\sigma}$ is
a standard Gaussian

Warmup

If X is a r.v. with pdf $f_X(x)$, $a > 0$, $b \in \mathbb{R}$

$Y = aX + b$. What is the pdf of Y ?

Warmup

If X is a r.v. with pdf $f_X(x)$, $a > 0$, $b \in \mathbb{R}$

$Y = aX + b$. What is the pdf of Y ?

$$F_Y(y) = \Pr[Y \leq y] = \Pr[aX + b \leq y] = \Pr[X \leq \frac{y-b}{a}] \\ = F_X\left(\frac{y-b}{a}\right).$$

$$f_Y(y) = \frac{dF_Y(y)}{dy} = \frac{dF_X\left(\frac{y-b}{a}\right)}{dy} = f_X\left(\frac{y-b}{a}\right) \cdot \frac{1}{a}$$

Theorem:

1. If $X \sim N(0,1)$ then $Y = \sigma X + \mu$
is dist. $N(\mu, \sigma^2)$.

2. If $Y \sim N(\mu, \sigma^2)$ then $X = \frac{Y - \mu}{\sigma}$
is a standard Gaussian

Warmup

If $Y = aX + b$

$$f_Y(y) = f_X\left(\frac{y-b}{a}\right) \frac{1}{|a|}$$

Proof: 1. $f_Y(y) = f_X\left(\frac{y-\mu}{\sigma}\right) \frac{1}{\sigma} = \frac{1}{\sigma} \frac{1}{\sqrt{2\pi}} e^{-\frac{(y-\mu)^2}{2\sigma^2}} \leftarrow N(\mu, \sigma^2)$

2. $f_X(x) = f_Y(\sigma x + \mu) \cdot \sigma = \frac{\sigma}{\sqrt{2\pi\sigma^2}} \cdot e^{-\frac{(\sigma x + \mu - \mu)^2}{2\sigma^2}}$
 $= \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \leftarrow N(0,1)$

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Expectation and Variance of Gaussian

$$X \sim N(0, 1)$$

$$f_x(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$$

$$E[X] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x \cdot e^{-x^2/2} dx$$

Expectation and Variance of Gaussian

$$X \sim N(0,1) \quad f_x(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$$

$$E[X] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x \cdot e^{-x^2/2} dx$$

$$= \frac{1}{\sqrt{2\pi}} \left(\int_{-\infty}^0 x \cdot e^{-x^2/2} dx + \int_0^{\infty} x \cdot e^{-x^2/2} dx \right) = 0$$


↔
cancel each other.

Expectation and Variance of Gaussian

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 cancel each other.

$$\text{Var}(X) = E[X^2] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x^2 \cdot e^{-x^2/2} dx$$

Expectation and Variance of Gaussian

$$X \sim N(0,1) \quad f_x(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$$

$$\begin{aligned} E[X] &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x \cdot e^{-x^2/2} dx \\ &= \frac{1}{\sqrt{2\pi}} \left(\int_{-\infty}^0 x \cdot e^{-x^2/2} dx + \int_0^{\infty} x \cdot e^{-x^2/2} dx \right) = 0 \end{aligned}$$

↖ ↗
cancel each other.

$$\begin{aligned} \text{Var}(X) = E[X^2] &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x^2 \cdot e^{-x^2/2} dx \\ &= \frac{1}{\sqrt{2\pi}} \left(\int_{-\infty}^0 (-x) \cdot (-x \cdot e^{-x^2/2}) dx + \int_0^{\infty} x \cdot (x \cdot e^{-x^2/2}) dx \right) \\ &= \frac{1}{\sqrt{2\pi}} \left(\int_{-\infty}^0 (-x) \cdot e^{-x^2/2} dx + \int_0^{\infty} x \cdot e^{-x^2/2} dx \right) = 0 + 1 = 1. \end{aligned}$$

Theorem: let $X \sim N(\mu, \sigma^2)$ then

$$E[X] = \mu$$

$$\text{Var}(X) = \sigma^2$$

Proof:

Theorem: Let $X \sim N(\mu, \sigma^2)$ then

$$E[X] = \mu$$

$$\text{Var}(X) = \sigma^2$$

Proof:

By previous theorems $Y = \frac{X - \mu}{\sigma}$ is standard

Gaussian and has mean 0, variance 1.

$$E[X] = E[\sigma Y + \mu] = \sigma E[Y] + \mu = 0 + \mu$$

$$\text{Var}[X] = \text{Var}[\sigma Y + \mu] = \sigma^2 \text{Var}[Y] = \sigma^2$$



Two Independent Gaussians

If X, Y are independent standard Gaussians then

their joint dist is $f_{X,Y}(x,y) = \frac{1}{2\pi} \cdot e^{-(x^2+y^2)/2}$

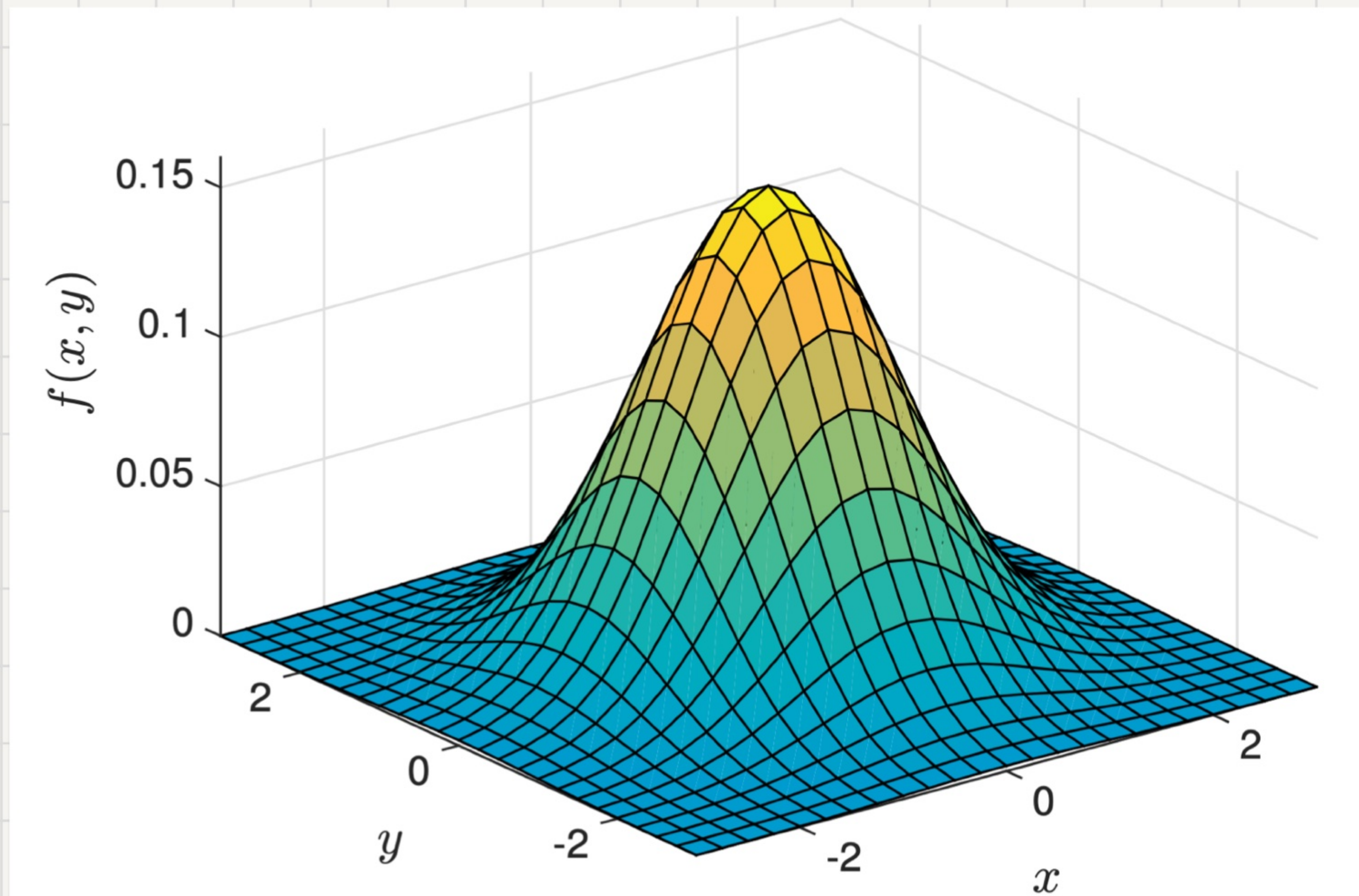
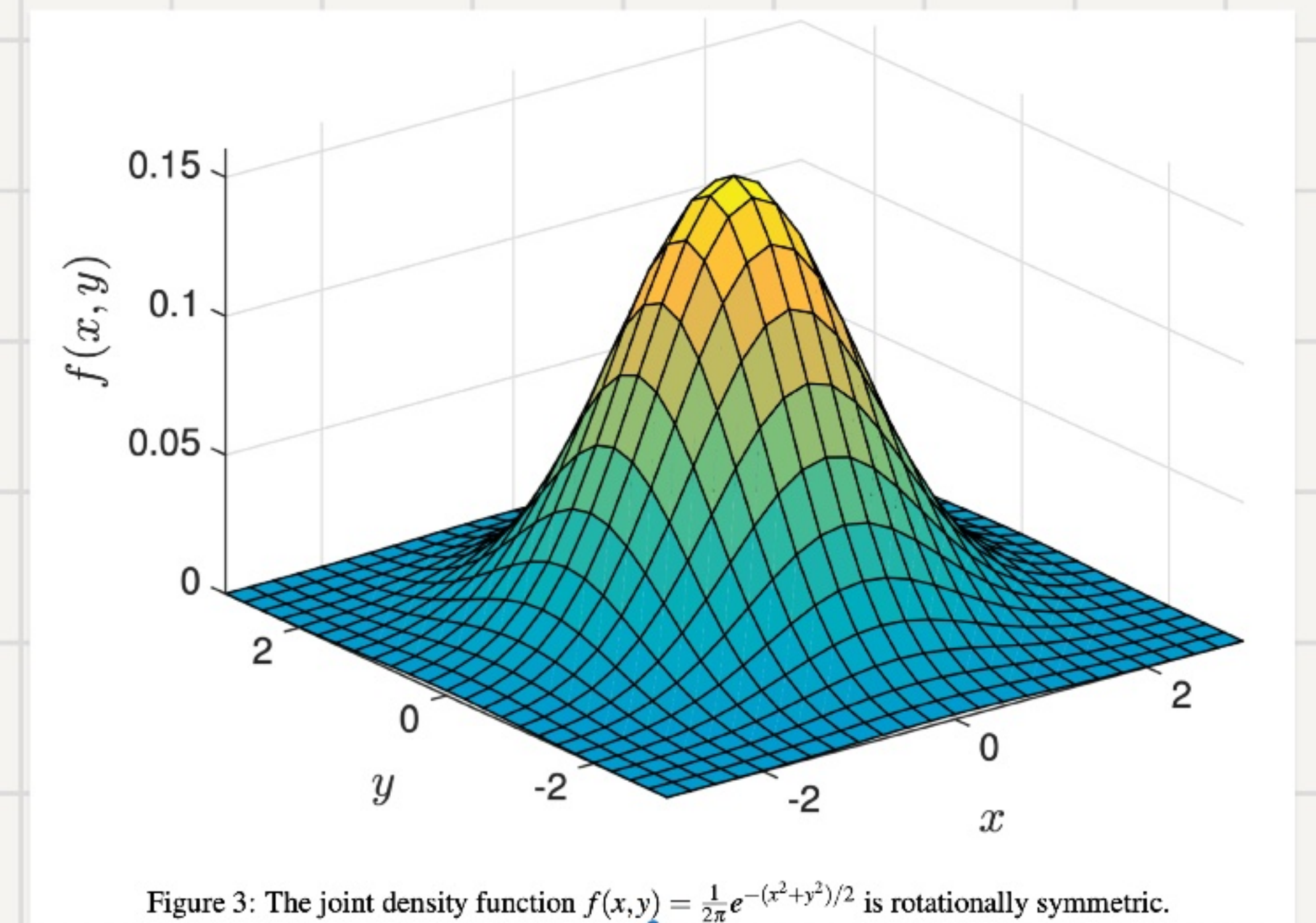


Figure 3: The joint density function $f(x,y) = \frac{1}{2\pi} e^{-(x^2+y^2)/2}$ is rotationally symmetric.

Two Independent Gaussians

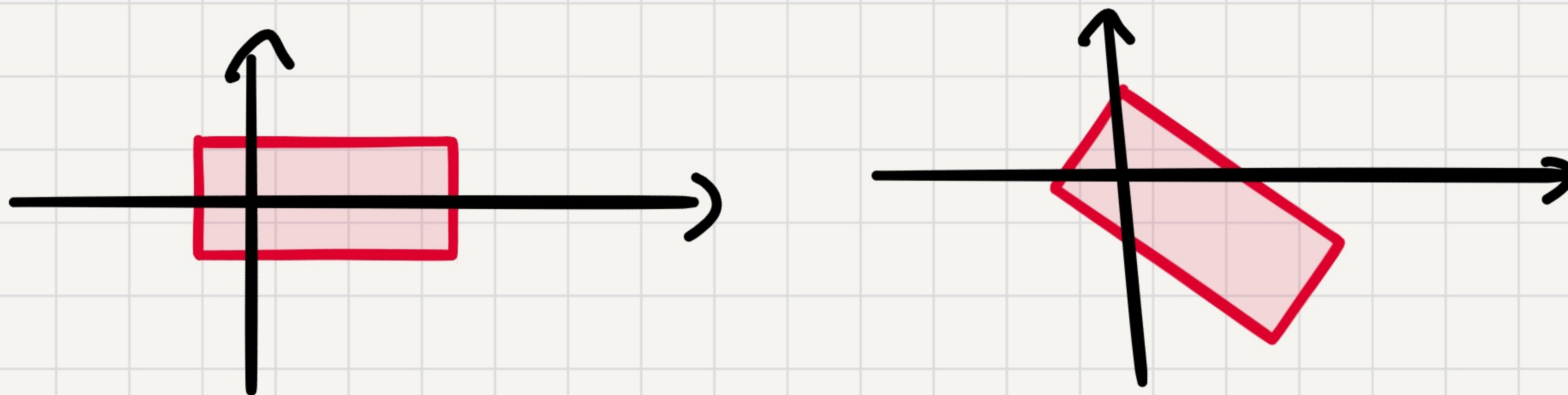
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their joint dist is $f_{X,Y}(x,y) = \frac{1}{2\pi} \cdot e^{-(x^2+y^2)/2}$



Observation: The joint distribution is rotation symmetric -
depends only on radius, not on angle.

e.g.,



the two events occur with the same probability

Sum of Independent Gaussians is a Gaussian

Theorem: If $X \sim N(0,1)$ and $Y \sim N(0,1)$ are independent

then $X + Y \sim N(0,2)$.

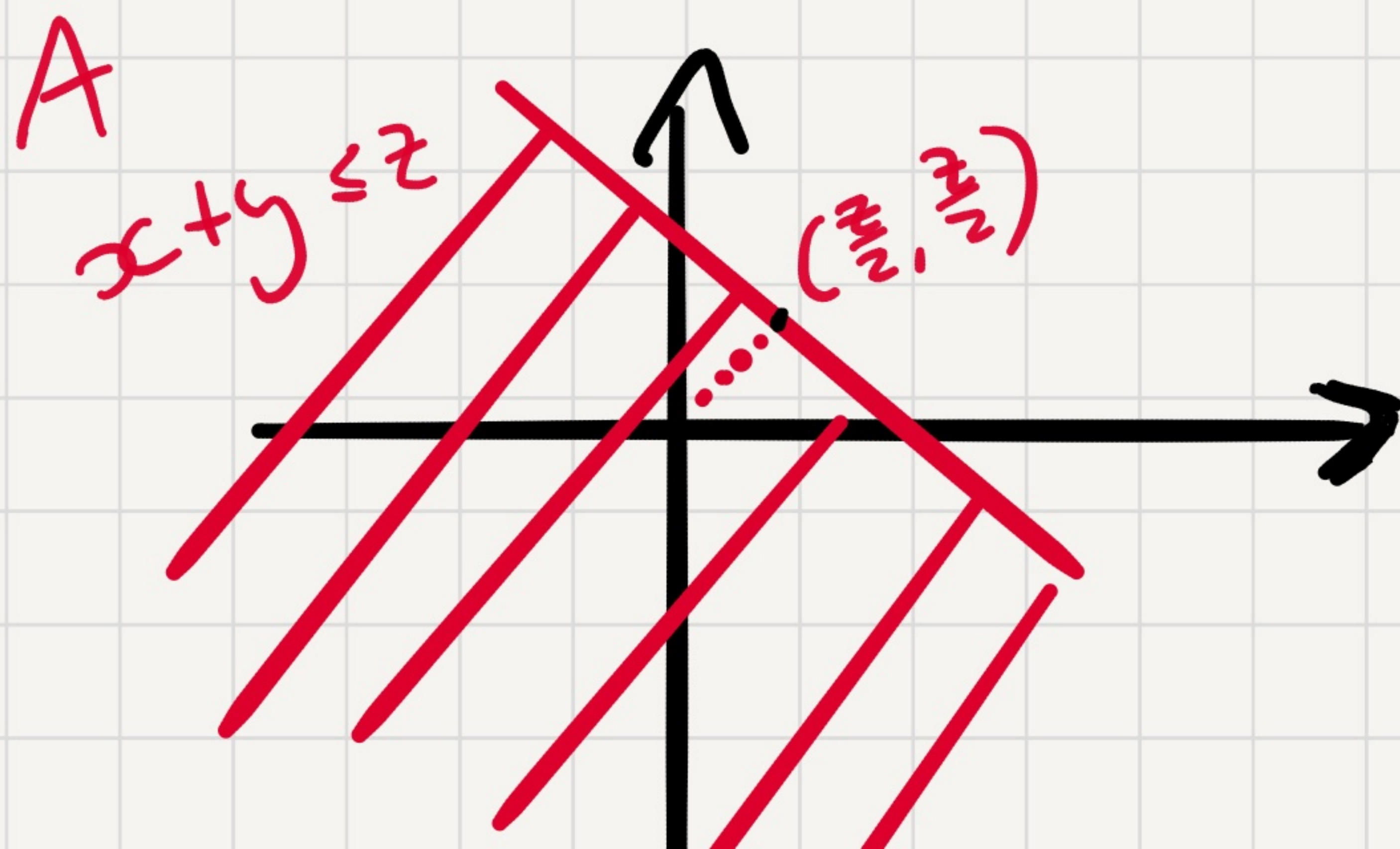
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Theorem: If $X \sim N(0,1)$ and $Y \sim N(0,1)$ are independent

then $X+Y \sim N(0,2)$

Proof: Let $Z = X+Y$

$$\begin{aligned} P_r [Z \leq z] &= P_r [X+Y \leq z] \\ &= P_r [(X,Y) \in A] \end{aligned}$$



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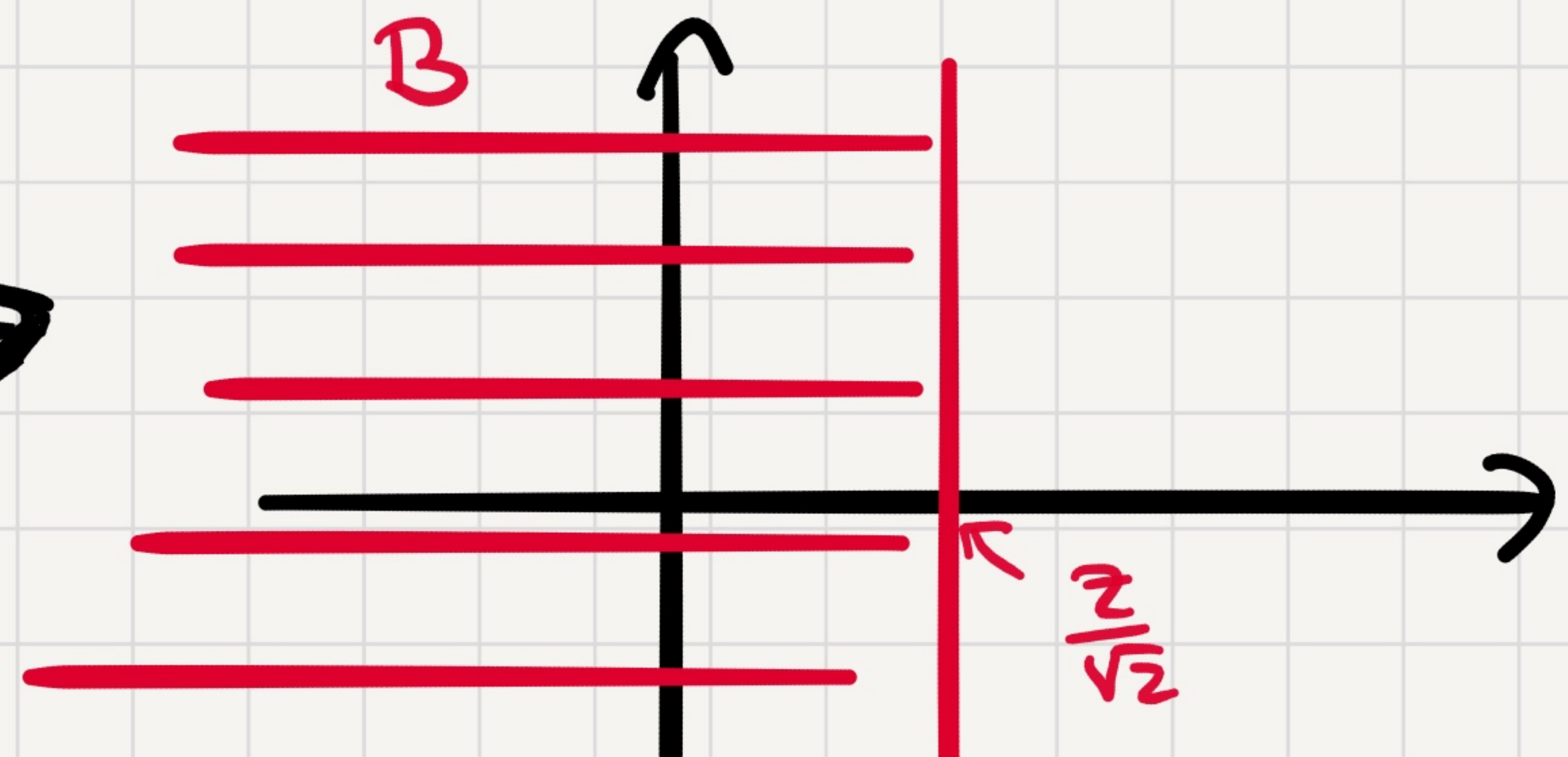
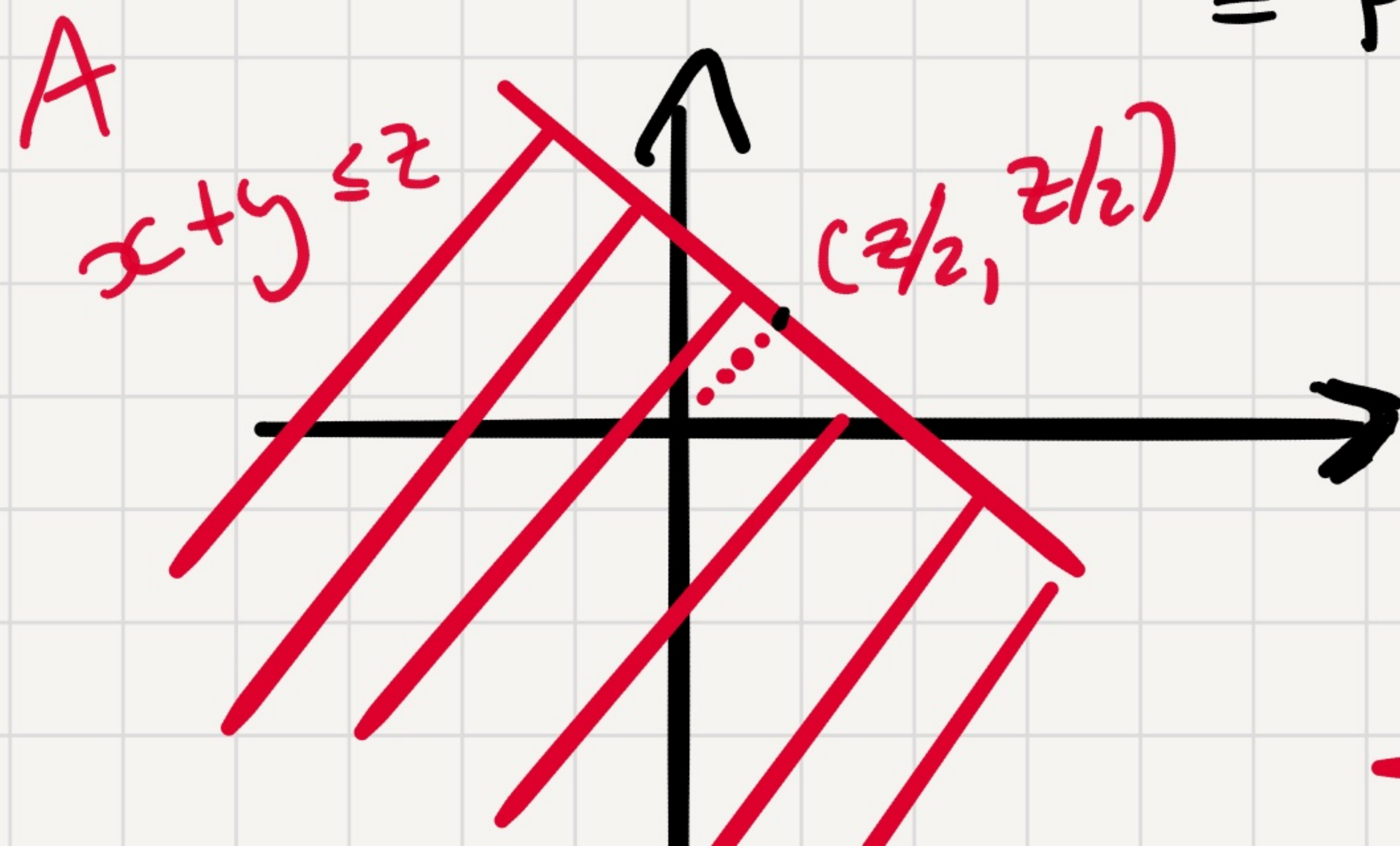
then $X+Y \sim N(0,2)$

Proof: Let $Z = X+Y$

$$P_Y [Z \leq z] = P_Y [X+Y \leq z]$$

$$= P_Y [(X,Y) \in A]$$

$$= P_Y [(X,Y) \in B] = P_Y [X \leq \frac{z}{\sqrt{2}}]$$



Sum of Independent Gaussians is a Gaussian

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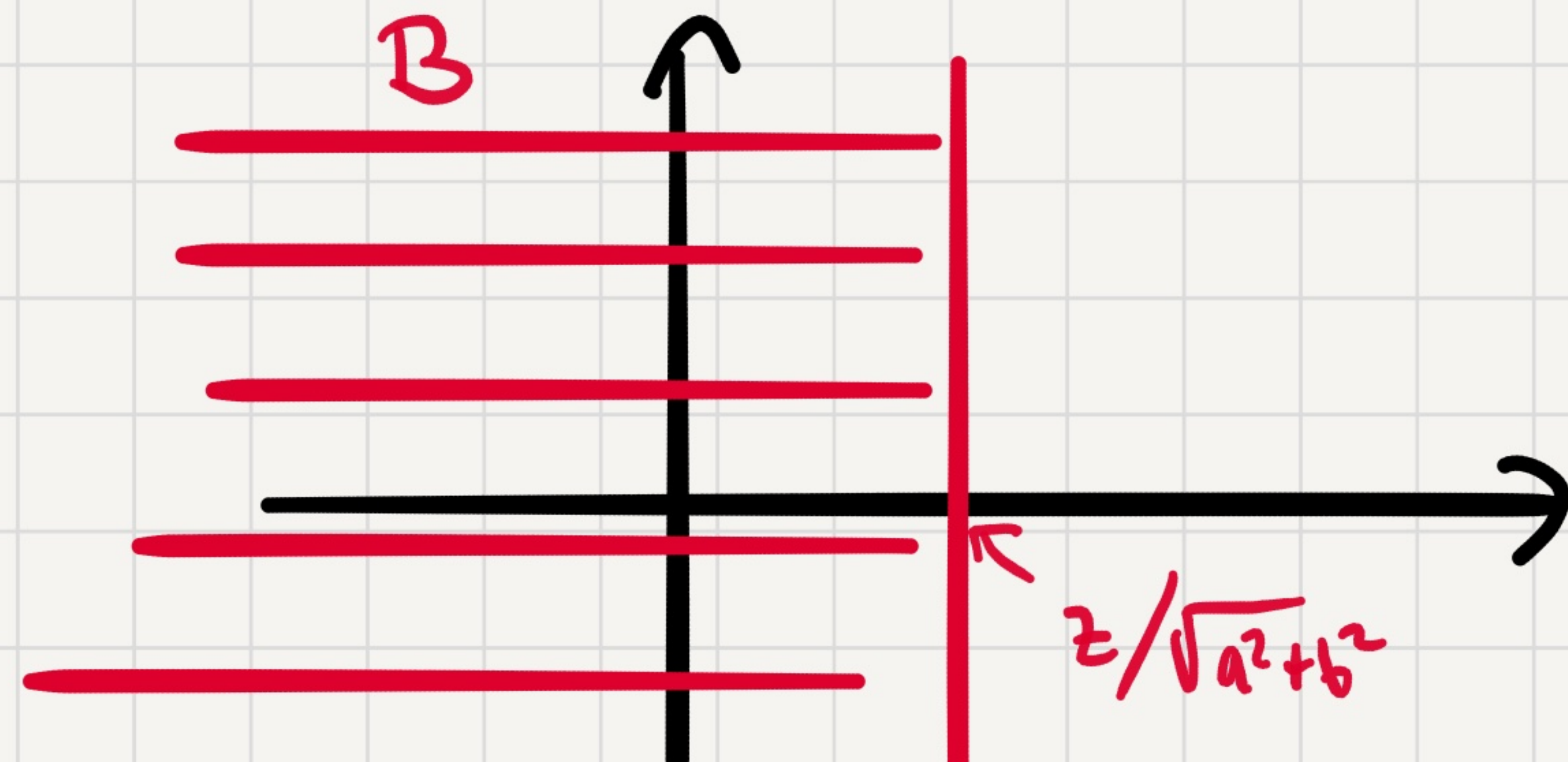
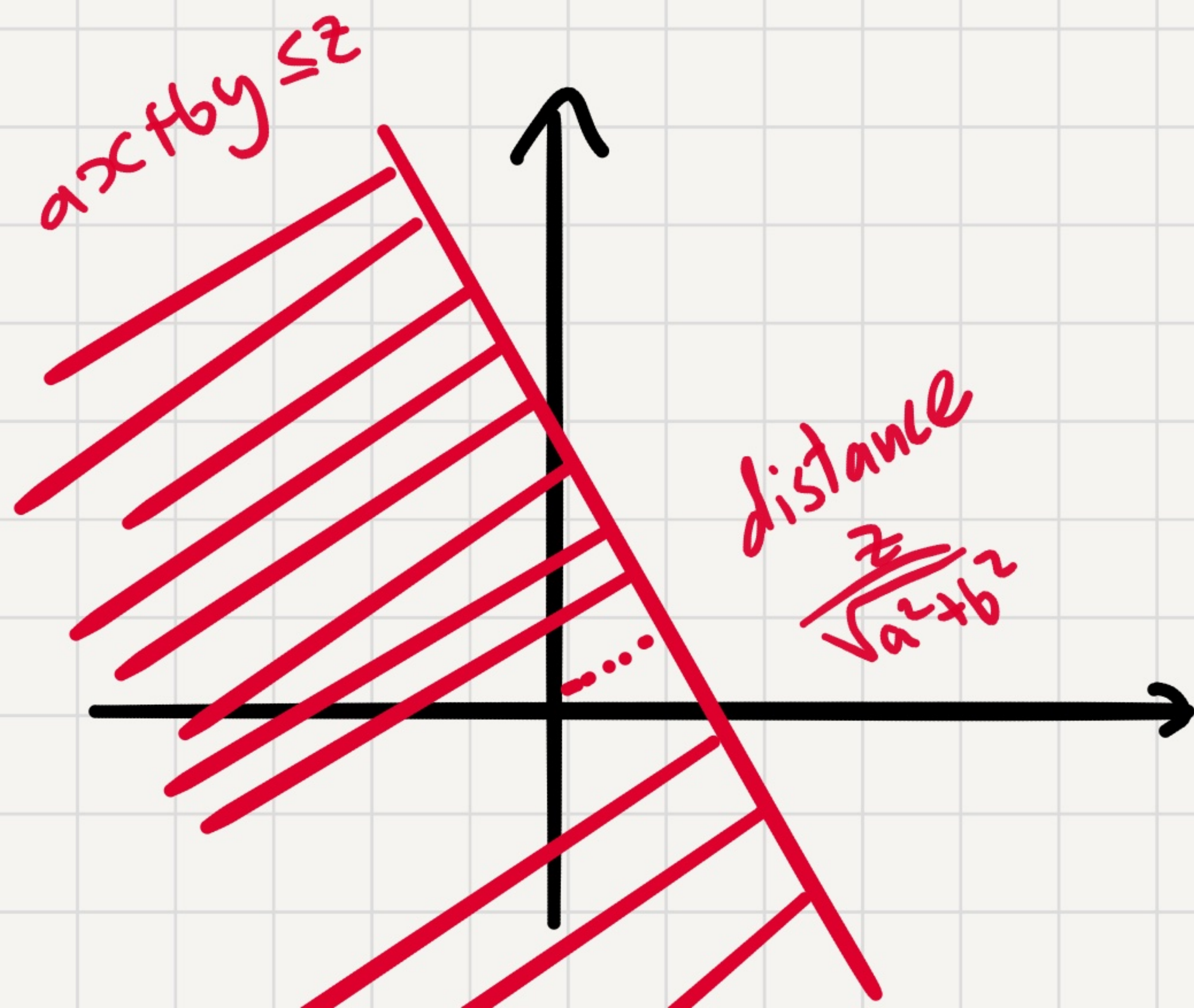
then $aX + bY \sim N(0, a^2 + b^2)$

Proof: Let $Z = aX + bY$

$$\Pr[Z \leq z] = \Pr[aX + bY \leq z]$$

$$= \Pr[(X, Y) \in A]$$

$$= \Pr[(X, Y) \in B] = \Pr\left[X \leq \frac{z}{\sqrt{a^2 + b^2}}\right]$$



Sum of Independent Gaussians is a Gaussian

Theorem: If $X \sim N(\mu_1, \sigma_1^2)$, $Y \sim N(\mu_2, \sigma_2^2)$
are indep. then $X+Y \sim N(\mu_1 + \mu_2, \sigma_1^2 + \sigma_2^2)$.

Sum of Independent Gaussians is a Gaussian

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are indep. then $X+Y \sim N(\mu_1 + \mu_2, \sigma_1^2 + \sigma_2^2)$.

Proof: $\hat{X} = \frac{X - \mu_1}{\sigma_1}$ $\hat{Y} = \frac{Y - \mu_2}{\sigma_2}$

\hat{X}, \hat{Y} indep. standard Gaussians

$$\begin{aligned} X+Y &= \sigma_1 \hat{X} + \mu_1 + \sigma_2 \hat{Y} + \mu_2 \\ &= (\mu_1 + \mu_2) + \underbrace{\sigma_1 \hat{X} + \sigma_2 \hat{Y}}_{N(0, \sigma_1^2 + \sigma_2^2)} \\ &\underbrace{\hspace{10em}}_{N(\mu_1 + \mu_2, \sigma_1^2 + \sigma_2^2)}. \end{aligned}$$

Recap

Weak Law of Large Numbers

If X_1, X_2, \dots are i.i.d.

with $E X_i = \mu$ $Var(X_i) = \sigma^2$

then for all $\epsilon > 0$

$$Pr \left[\left| \frac{X_1 + \dots + X_n}{n} - \mu \right| > \epsilon \right] \xrightarrow{n \rightarrow \infty} 0$$

The Central Limit Theorem

If X_1, X_2, \dots are independent

with $E X_i = \mu$ $\text{Var}(X_i) = \sigma^2$

then
$$\frac{X_1 + \dots + X_n - \mu n}{\sigma \cdot \sqrt{n}} \xrightarrow[n \rightarrow \infty]{} N(0, 1).$$

The Central Limit Theorem

If X_1, X_2, \dots are independent

with $E X_i = \mu$ $\text{Var}(X_i) = \sigma^2$

then

$$\frac{X_1 + \dots + X_n - \mu n}{\sigma \cdot \sqrt{n}} \xrightarrow{n \rightarrow \infty} N(0, 1).$$

Special Case:

If X_1, X_2, \dots are indep. with mean 0
variance 1

then
$$\frac{X_1 + \dots + X_n}{\sqrt{n}} \rightarrow N(0, 1)$$

The Central Limit Theorem

If X_1, X_2, \dots are independent

with $E X_i = \mu$ $\text{Var}(X_i) = \sigma^2$

then $\frac{X_1 + \dots + X_n - \mu n}{\sigma \cdot \sqrt{n}} \xrightarrow[n \rightarrow \infty]{} N(0, 1)$.

Sanity Check:

What's the mean, var of $\frac{X_1 + \dots + X_n - \mu n}{\sigma \sqrt{n}}$?

Why does it make sense that $N(0,1)$ is
the limit?

X_1, X_2, \dots , indep. r.v.s with mean 0 and variance 1.

If we assume the limit $\frac{X_1 + \dots + X_n}{\sqrt{n}}$ exists, then

$$\frac{X_1 + X_3 + \dots + X_{2n-1}}{\sqrt{n}} \xrightarrow{n \rightarrow \infty} Z_{\text{odd}}$$

$$\frac{X_2 + X_4 + \dots + X_{2n}}{\sqrt{n}} \xrightarrow{n \rightarrow \infty} Z_{\text{even}}$$

$$\frac{X_1 + \dots + X_{2n}}{\sqrt{2n}} \xrightarrow{n \rightarrow \infty} Z$$

Why does it make sense that $N(0,1)$ is the limit?

X_1, X_2, \dots r.v.s with mean 0 and variance 1.

If we assume the limit $\frac{X_1 + \dots + X_n}{\sqrt{n}}$ exists, then

$$\frac{X_1 + X_3 + \dots + X_{2n-1}}{\sqrt{n}}$$

$\xrightarrow[n \rightarrow \infty]$

Z_{odd}

$Z_{\text{odd}}, Z_{\text{even}}, Z$ have the same distribution

$$\frac{X_2 + X_4 + \dots + X_{2n}}{\sqrt{n}}$$

$\xrightarrow[n \rightarrow \infty]$

Z_{even}

$Z_{\text{odd}}, Z_{\text{even}}$ indep.

$$\frac{X_1 + \dots + X_{2n}}{\sqrt{2n}}$$

$\xrightarrow[n \rightarrow \infty]$

Z

$$Z = \frac{Z_{\text{odd}} + Z_{\text{even}}}{\sqrt{2}}$$

Gaussian dist. satisfies that!

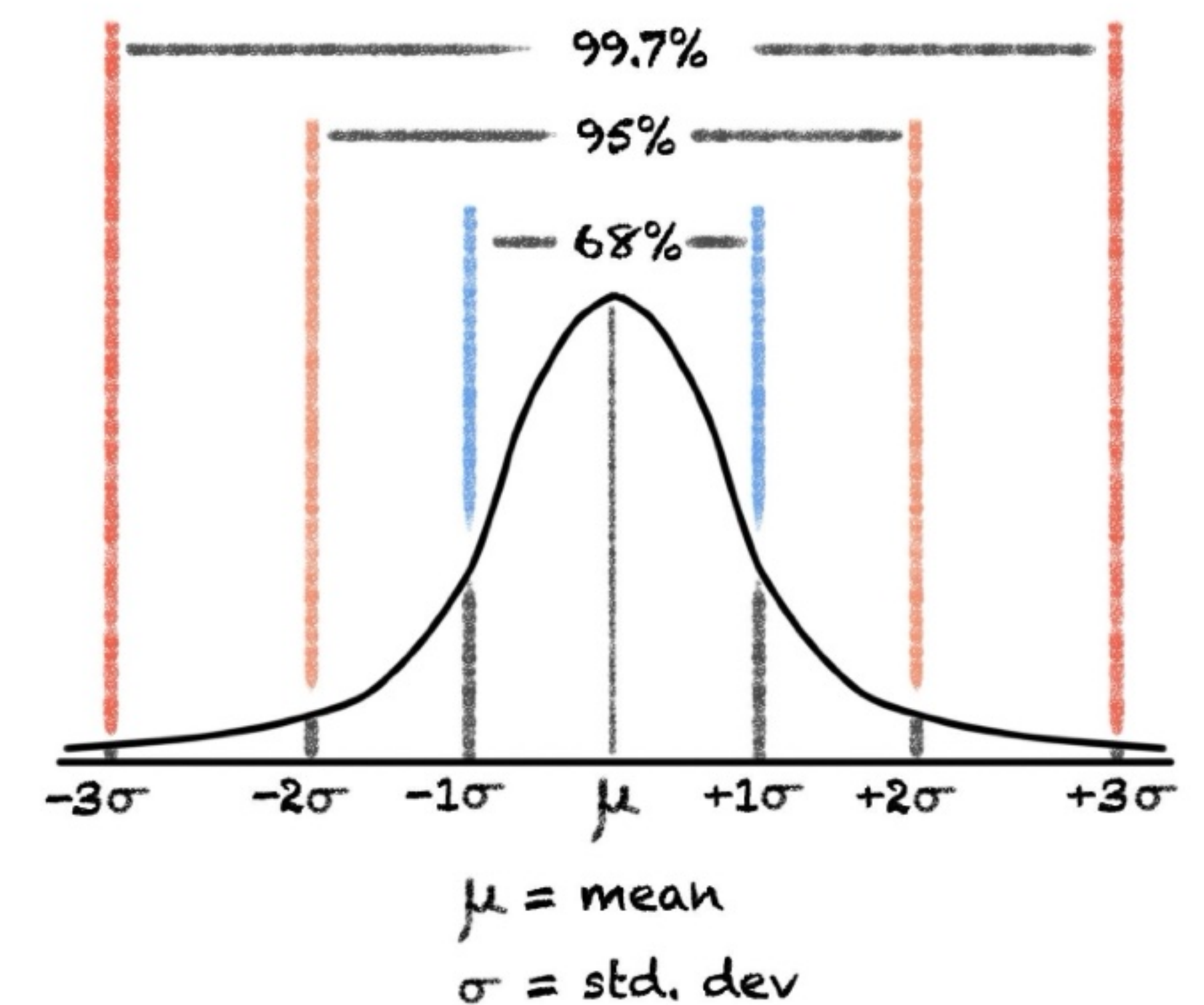
The Central Limit Theorem

If X_1, X_2, \dots are i.i.d.

with $E X_i = \mu$ $\text{Var}(X_i) = \sigma^2$

then $\frac{X_1 + \dots + X_n - \mu n}{\sigma \cdot \sqrt{n}} \xrightarrow{n \rightarrow \infty} N(0,1)$.

Normal Distribution



Neil Kakkar

Example:

$X \sim \text{Bin}(n, p)$, $E[X] = np$ $\sigma(X) = \sqrt{np(1-p)}$

$$\Pr[|X - E[X]| \leq k \cdot \sigma(X)] \xrightarrow{n \rightarrow \infty} \Pr[|Z| \leq k] = \begin{cases} 0.68\dots, & k=1 \\ 0.95\dots, & k=2 \\ 0.997\dots, & k=3 \end{cases}$$

$Z \sim N(0,1)$

More Scaling and Shifting

We saw if $Y = aX + b$

$$f_Y(y) = f_X\left(\frac{y-b}{a}\right) \cdot \frac{1}{|a|}$$

Example 1: If $X \sim U([0, 1])$

$$Y = 2X$$

What's the dist. of Y ?

More Scaling and Shifting

We saw if $Y = aX + b$

$$f_Y(y) = f_X\left(\frac{y-b}{a}\right) \cdot \frac{1}{|a|}$$

Example 1: If $X \sim U([0, 1])$

$Y = lX$. What's the dist. of Y ?

$$d_Y(y) = \begin{cases} 1/l, & \text{if } 0 \leq y \leq l \\ 0, & \text{o.w.} \end{cases} \quad U([0, l])$$

$Y = lX + b$? $U([b, b+l])$

So all uniform r.v.s are similar...

More Scaling and Shifting

We saw if $Y = aX + b$

$$f_Y(y) = f_X\left(\frac{y-b}{a}\right) \cdot \frac{1}{|a|}$$

Example 2: If $X \sim \text{Exp}(1)$

$$Y = aX$$

What's the dist. of Y ?

More Scaling and Shifting

We saw if $Y = aX + b$

$$f_Y(y) = f_X\left(\frac{y-b}{a}\right) \cdot \frac{1}{|a|}$$

Example 2: If $X \sim \text{Exp}(1)$

$$Y = aX$$

What's the dist. of Y ?

$$d_Y(y) = \frac{1}{a} \cdot e^{-y/a}$$

$$\text{Exp}(1/a)$$

$$\therefore Y = \frac{X}{\lambda}$$

is $\text{Exp}(\lambda)$.

So all exponential r.v.s are similar...

Minimum of Two Exponentials

Let $X \sim \text{Exp}(\lambda)$, $Y \sim \text{Exp}(\mu)$ independent.

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$$\Pr[Z \geq z] = \Pr[\min(X, Y) \geq z]$$

$$= \Pr[X \geq z, Y \geq z]$$

$$= \Pr[X \geq z] \Pr[Y \geq z]$$

$$= e^{-\lambda z} \cdot e^{-\mu z} = e^{-(\lambda + \mu)z}$$

$$\uparrow$$
$$Z \sim \text{Exp}(\lambda + \mu)$$

Maximum of Two Exponentials

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Calculate $E[Z] = ?$

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Trick: $Z = X + Y - \min(X, Y)$.

$$\begin{aligned} E[Z] &= E[X] + E[Y] - E[\min(X, Y)] \\ &= \frac{1}{\lambda} + \frac{1}{\mu} - \frac{1}{\lambda + \mu} \end{aligned}$$

Summary

- The Gaussian Distribution $N(\mu, \sigma^2)$

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

- $N(\mu, \sigma^2)$ is a scale & shift of $N(0, 1)$.
- Joint dist. of two independent standard Gaussians is rotation symmetric.
- Sums of two independent Gaussians is a Gaussian.
- Central limit theorem:

if x_1, x_2, \dots are independent w. mean μ
variance σ^2 then

$$\frac{x_1 + \dots + x_n - \mu n}{\sigma\sqrt{n}} \xrightarrow{n \rightarrow \infty} N(0, 1)$$