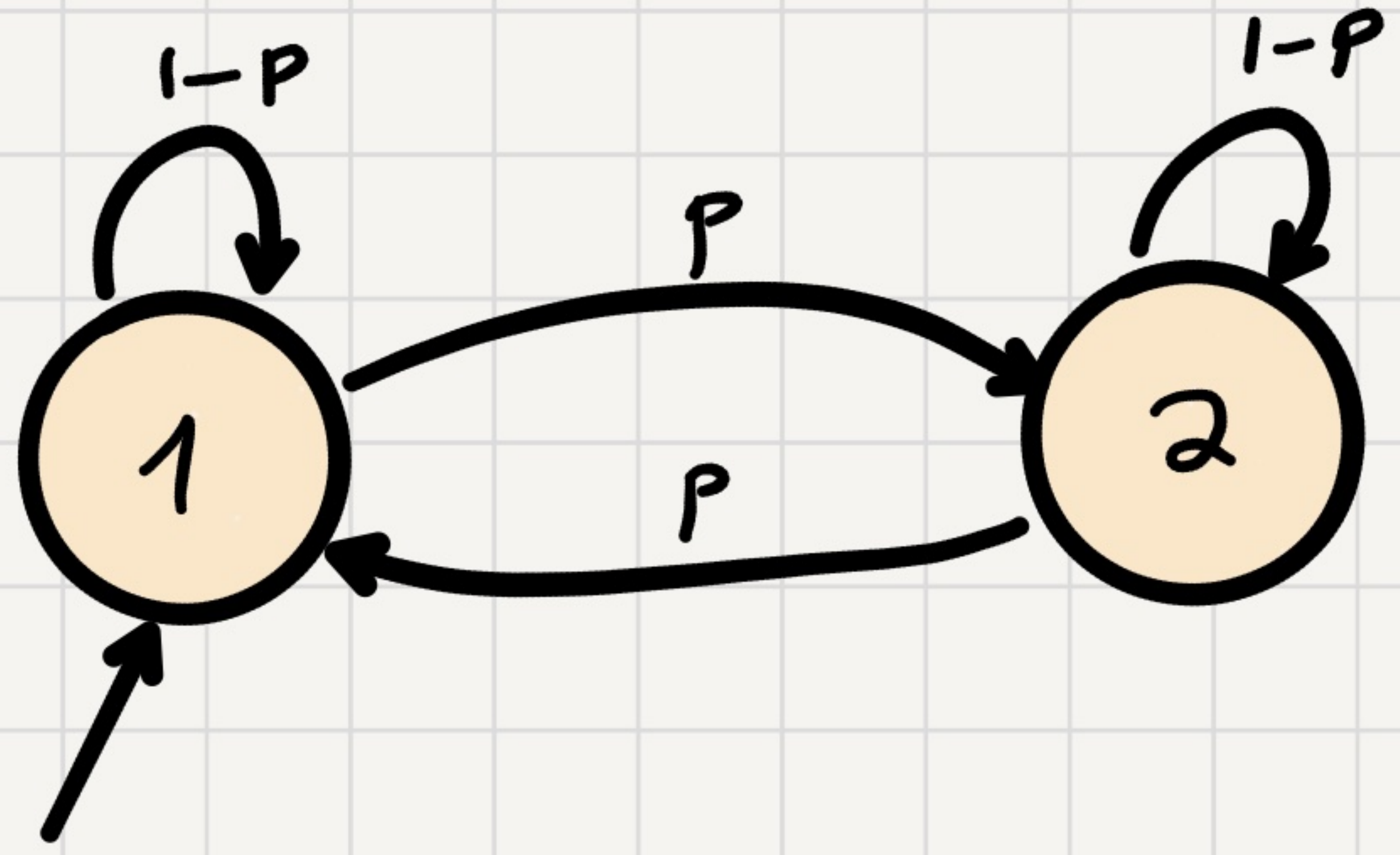
Lecture 26

Markov Chains

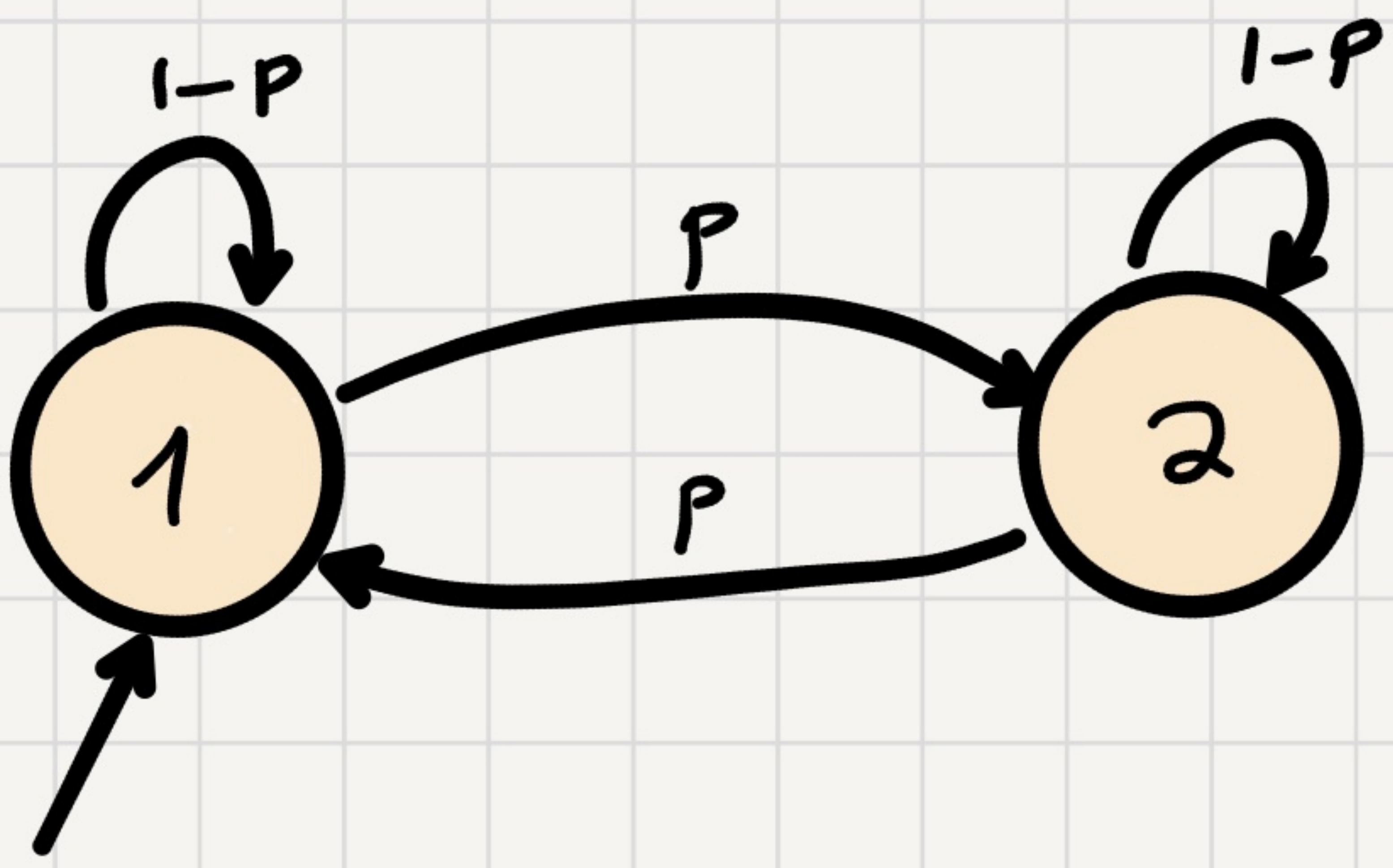
Example



- Start at 1.
- Every step, you stay in place w.p. $1-p$

  or move to the other state w.p. p.

Example



- Start at 1.
- Every step, you stay in place w.p. $1-p$

  or move to the other state w.p. p.

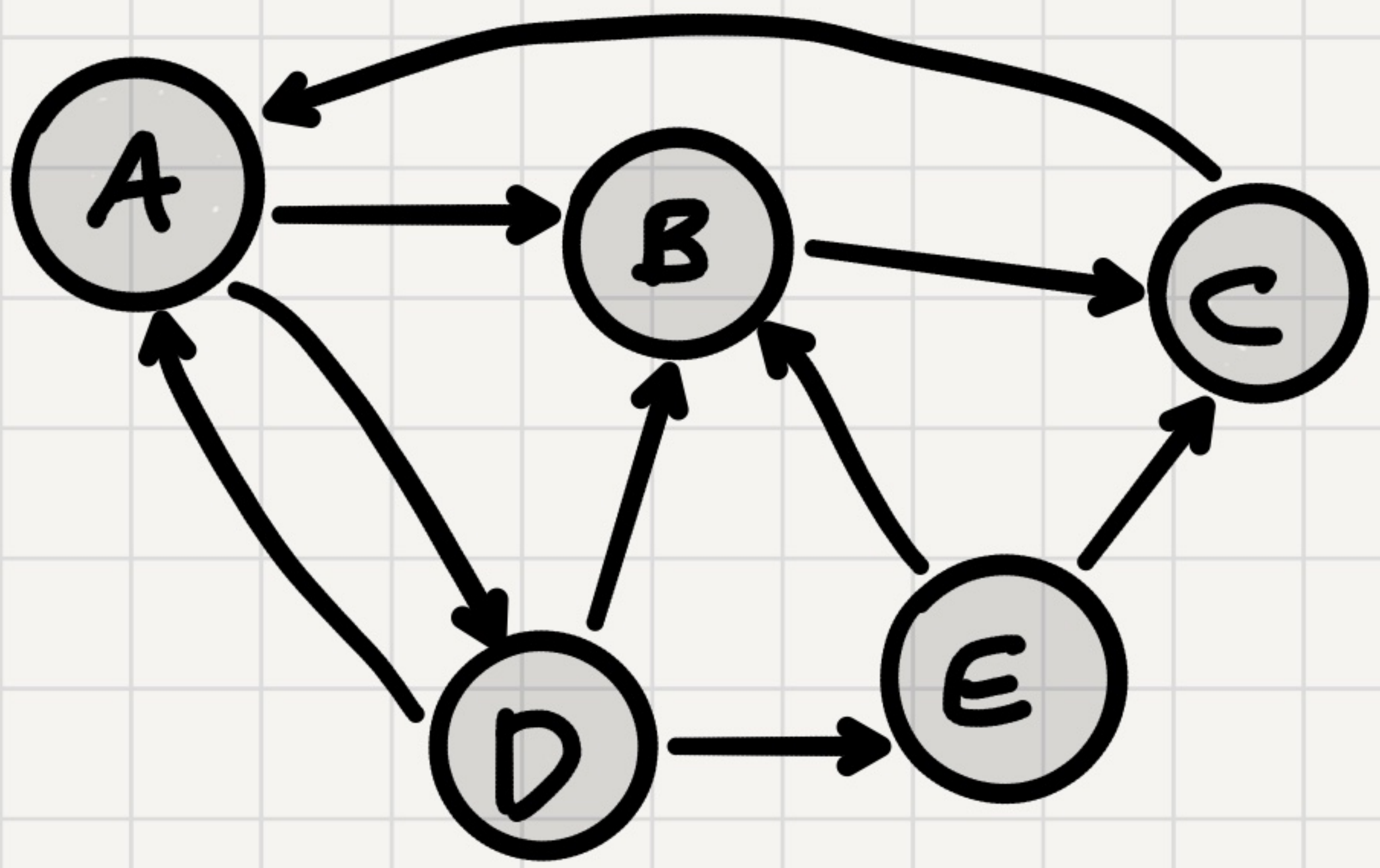
$p=0.2$, a sequence might be:

1 1 1 1 2 2 2 2 2 1 1 1 1 2 2 2 2 2 2 1 1 1

$p=0.5$,

1 2 2 2 1 2 1 1 2 1 2 ...

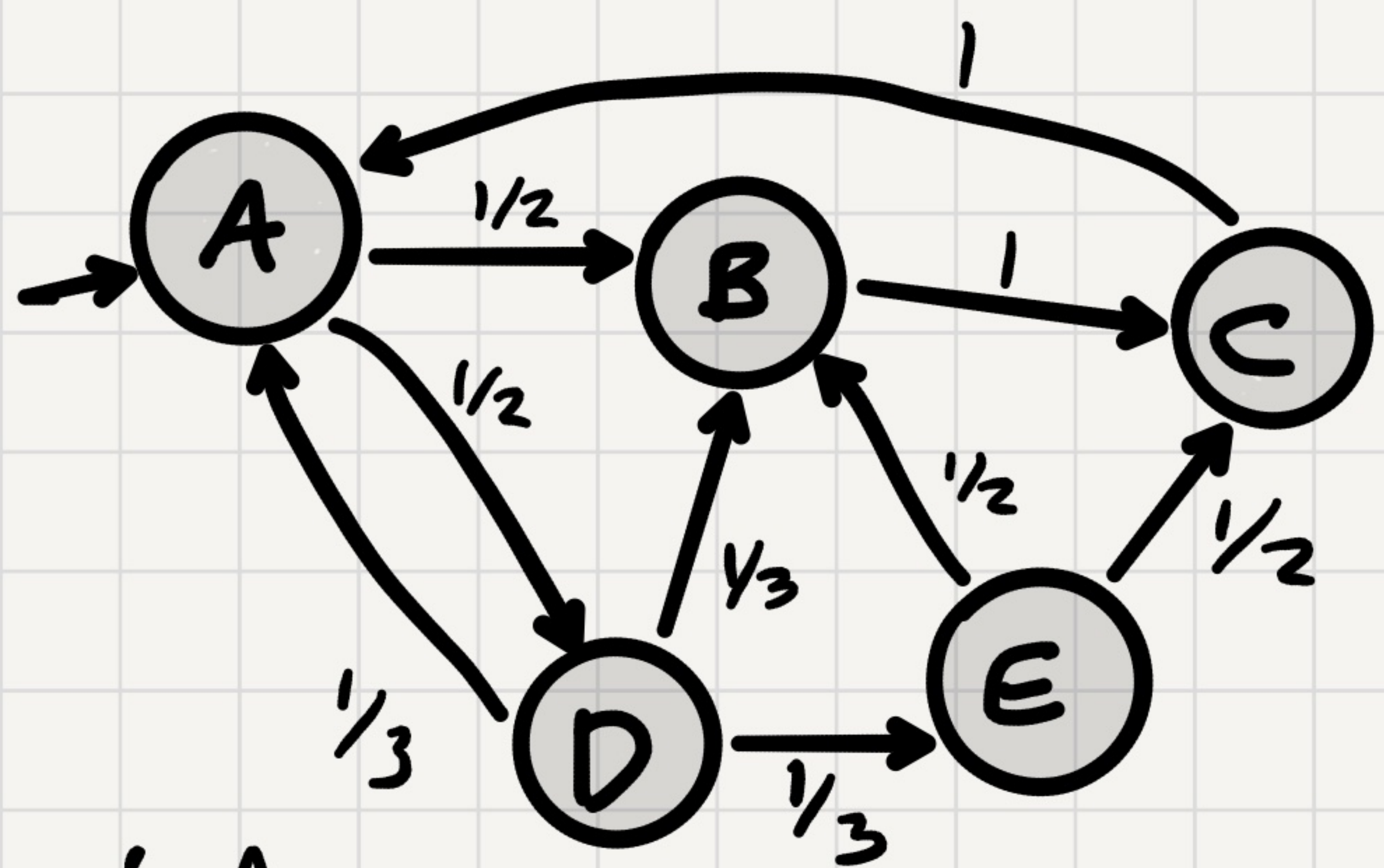
Another Example



At each step, follow one of the edges of the current state, with equal prob.

Can model links in the web.

Another Example



Start at A.

At each step, follow one of the edges of the current state, with equal prob.

Simulation:

A B C A B C ABCADADBC ABCADADEB ...

Markov Chain – Definition

Ingredients:

- A finite set of states $\mathcal{X} = \{1, 2, 3, \ldots, K\}$
- An initial prob. dist. π_0 on $\mathcal{X}$.
- Transition probabilities $P(i,j)$ for $i, j \in \mathcal{X}$

(the probability to move from i to j in one step).

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$$X_0, X_1, X_2, \ldots, X_n, \ldots$$

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- X_0 is sampled from π_0, i.e., $\Pr[X_0 = i] = \pi_0(i) \ \forall i \in \mathcal{X}$
- $\forall n \geq 0 \ \forall i_0, \dots, i_{n-1}, i, j \quad \Pr[X_{n+1} = j \mid X_n = i, X_{n-1} = i_{n-1}, \dots, X_0 = i_0]$

$$= \Pr[X_{n+1} = j \mid X_n = i] = P(i,j)$$

the only thing we remember about the past is the current state!

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- $\forall n \geq 0 \ \forall i, j \qquad \Pr[X_{n+1} = j \mid X_n = i, X_{n-1}, \dots, X_0] = P(i,j)$

What's the distribution of X_1?

What's the distribution of X_n?

What's the Distribution of X_1?

$$\Pr[X_0 = i] = \pi_0(i) \quad \forall i \in \{1, 2, \dots, k\}$$

$$\Pr[X_{n+1} = j \mid X_n = i, X_{n-1}, \dots, X_0] = P(i,j) \quad \forall i,j \in \{1, \dots, k\}$$

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$$\forall j \in \{1,\dots,k\} \quad \pi_1(j) = \Pr[X_1 = j] = \sum_{i=1}^{k} \Pr[X_0 = i, X_1 = j]$$

$$= \sum_{i=1}^{k} \Pr[X_0 = i] \Pr[X_1 = j \mid X_0 = i]$$

$$= \sum_{i=1}^{k} \pi_0(i) \cdot P(i,j)$$

If we think of π_0, π_1 as row vectors, P as a matrix

then $\pi_1 = \pi_0 \cdot P$.

What's the Distribution of X_n?

$$\Pr[X_0 = i] = \pi_0(i) \quad \forall i \in \{1,2,\ldots,k\}$$

$$\Pr[X_{n+1} = j \mid X_n = i, X_{n-1}, \ldots, X_0] = P(i,j) \quad \forall i,j \in \{1,\ldots,k\}$$

Let's denote the dist. of X_n by π_n

$$\forall j \in \{1,\ldots,k\} \quad \pi_n(j) = \Pr[X_n = j] = \sum_{i=1}^{k} \Pr[X_{n-1} = i, X_n = j]$$

$$= \sum_{i=1}^{k} \Pr[X_{n-1} = i] \cdot \Pr[X_n = j \mid X_{n-1} = i]$$

$$= \sum_{i=1}^{k} \pi_{n-1}(i) \cdot P(i,j)$$

<u>In vector-matrix form:</u> $\quad \pi_n = \pi_{n-1} \cdot P.$

What's the Distribution of X_n?

$$\Pr[X_0 = i] = \pi_0(i) \quad \forall i \in \{1, 2, \dots, k\}$$

$$\Pr[X_{n+1} = j \mid X_n = i, X_{n-1}, \dots, X_0] = P(i,j) \quad \forall i,j \in \{1, \dots, k\}$$

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$$= \sum_{i=1}^{k} \Pr[X_{n-1} = i] \cdot \Pr[X_n = j \mid X_{n-1} = i]$$

$$= \sum_{i=1}^{k} \pi_{n-1}(i) \cdot P(i,j)$$

<u>In vector-matrix form:</u> $\pi_n = \pi_{n-1} \cdot P$.

Thus, $\pi_n = \pi_{n-1} \cdot P = \pi_{n-2} \cdot P \cdot P = \dots = \pi_0 \cdot \underbrace{P \cdot P \cdots P}_{n \text{ times}} = \pi_0 \cdot P^n$.

Hitting Time – Example

Let's flip a coin w. heads prob. p until we get H.

How many flips on average?

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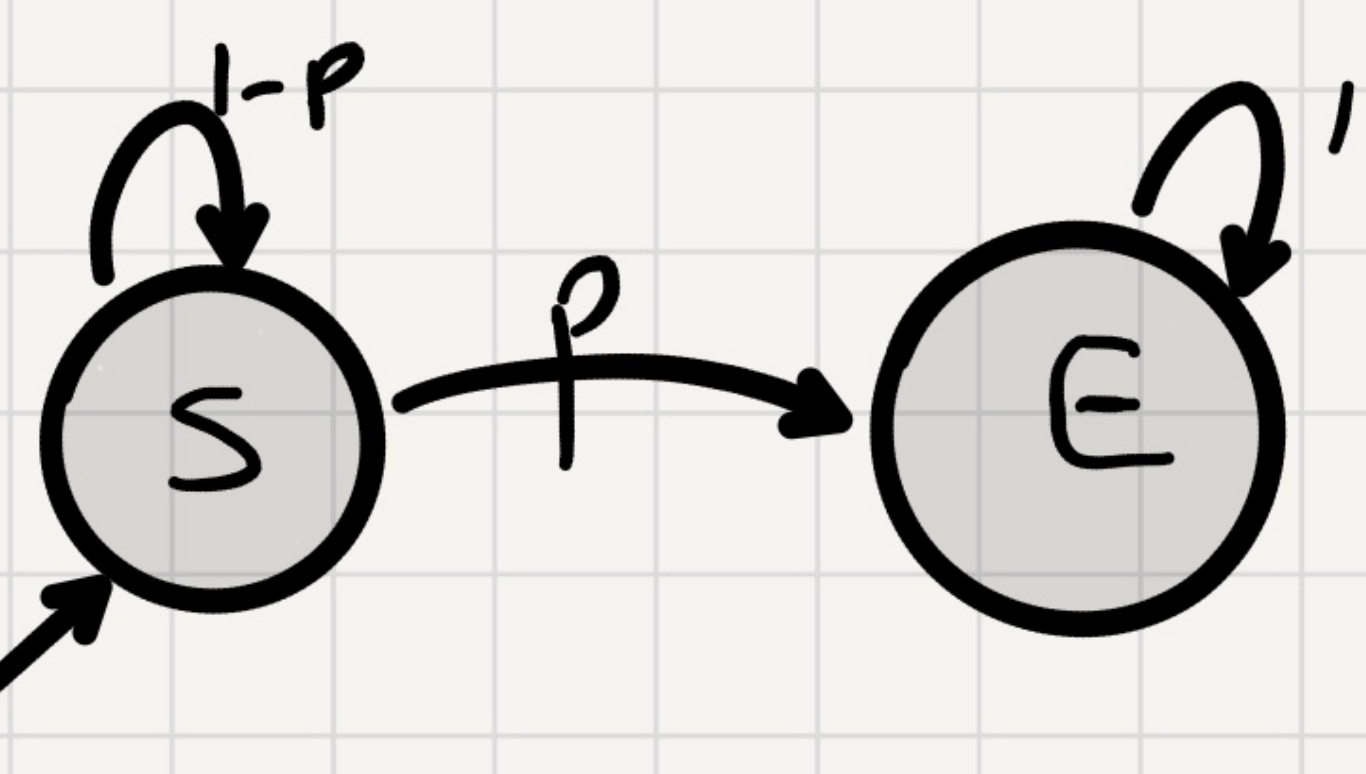
Let's flip a coin w. heads prob. p until we get H.

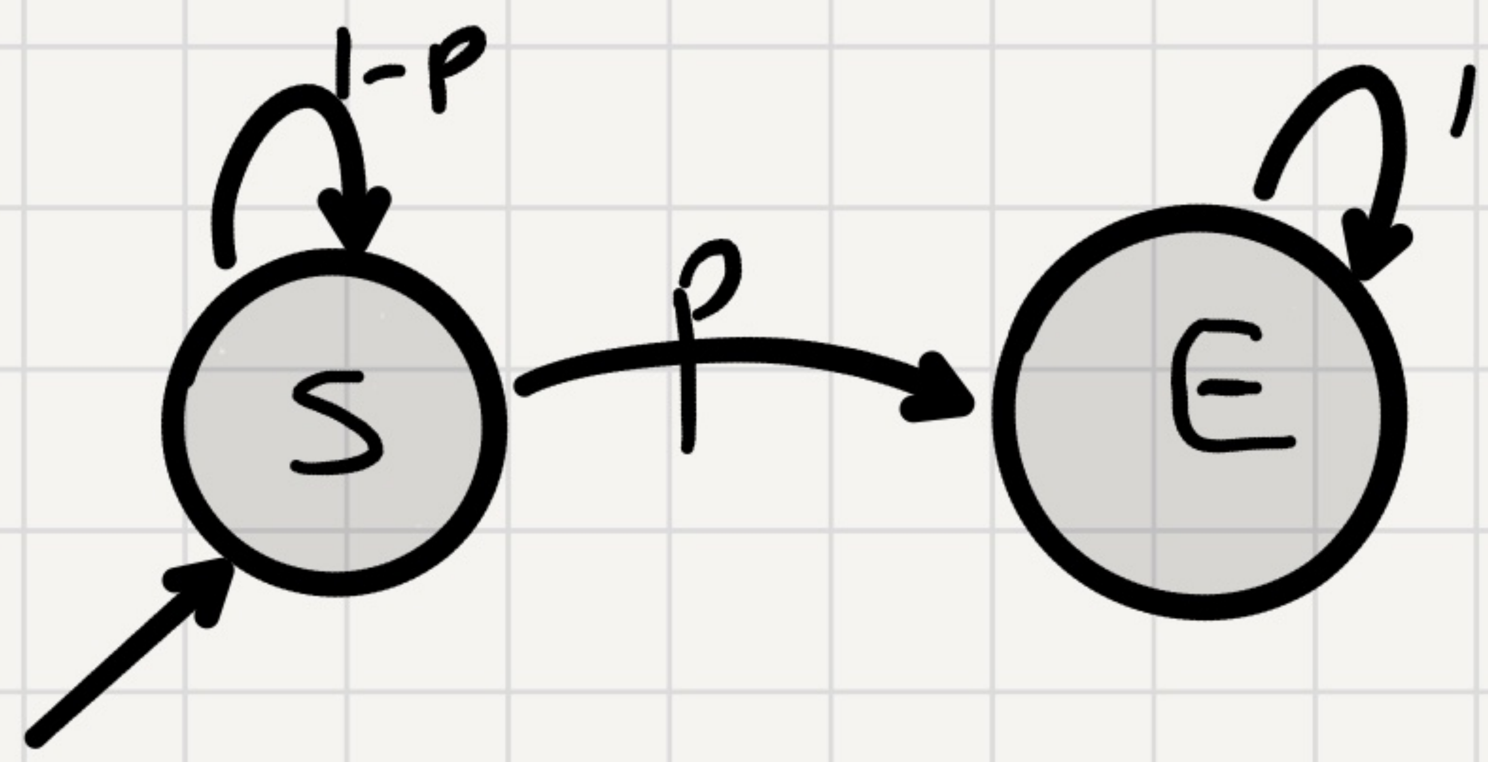
How many flips on average?

Let's model this as a Markov Chain.



Let $\beta(S)$ = average time until a MC starting from S would reach E.

Hitting Time - Example



Let $\beta(S)$ = average time until a MC starting from S would reach E.

What's true:

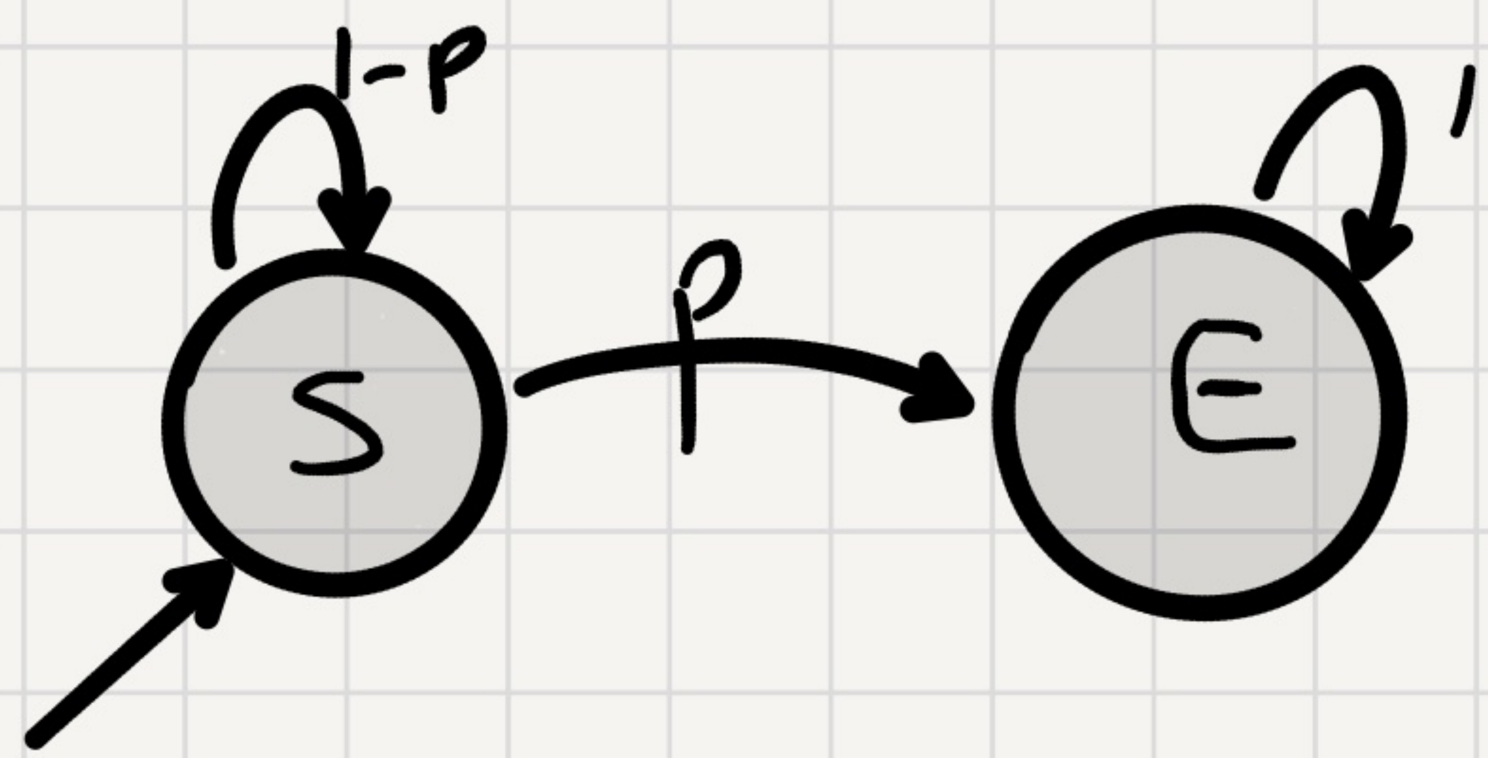
A. $\beta(S) \geq 1$.

B. From S we stay in S w.p. $1-p$.

C. From S we go to E w.p. p.

D. $\beta(S) = 1 + (1-p) \cdot \beta(S) + p \cdot 0$.

Hitting Time – Example



Let $\beta(S)$ = average time until a MC starting from S would reach E.

$$\boxed{\beta(S) = 1 + (1-p)\cdot\beta(S) + p\cdot 0.}$$

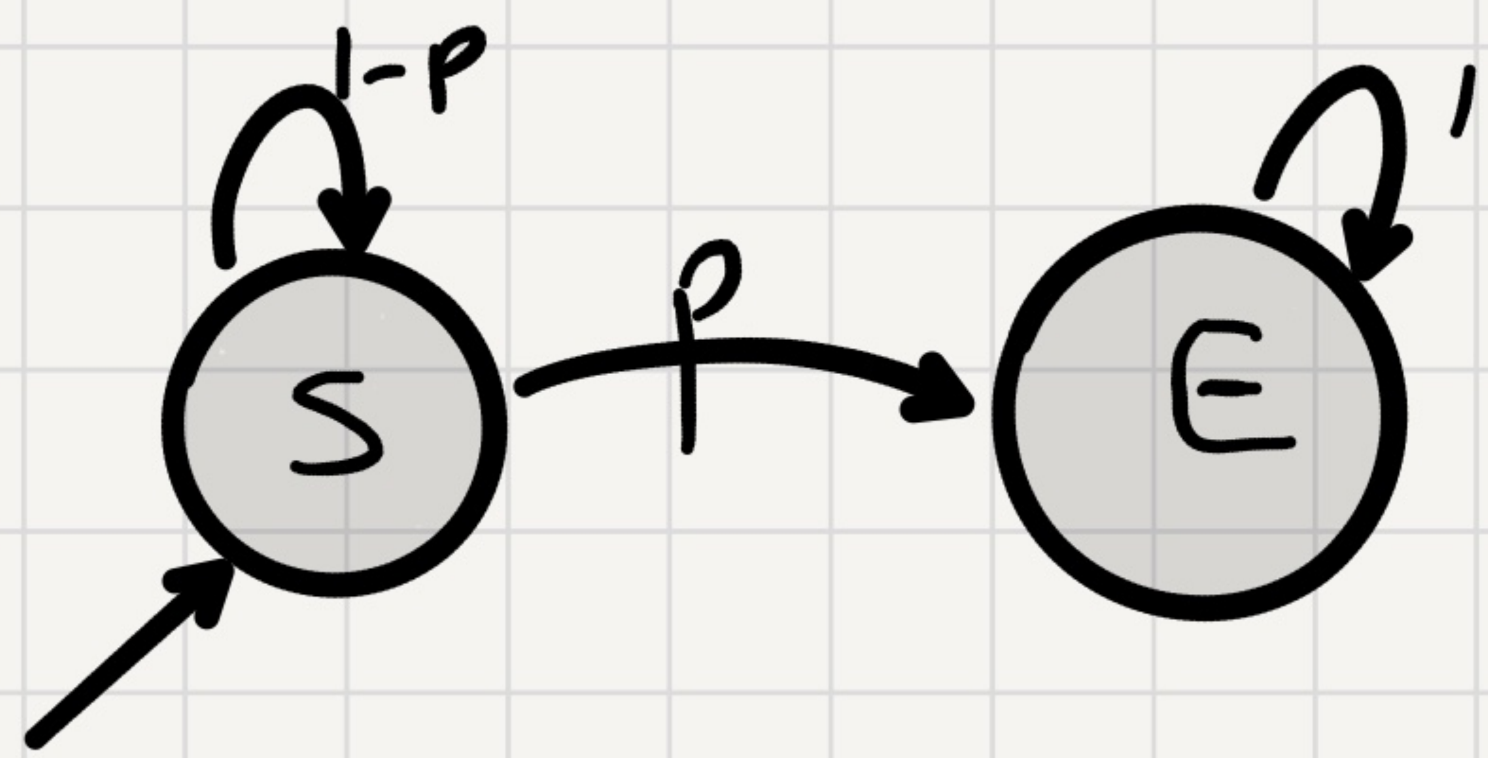
Justification:

We start at S and spend one step there.

Then, w.p. p we move to E and stop.

w.p. $1-p$ we stay in S, then the remaining time to hit E is again $\beta(S)$.

Hitting Time - Example



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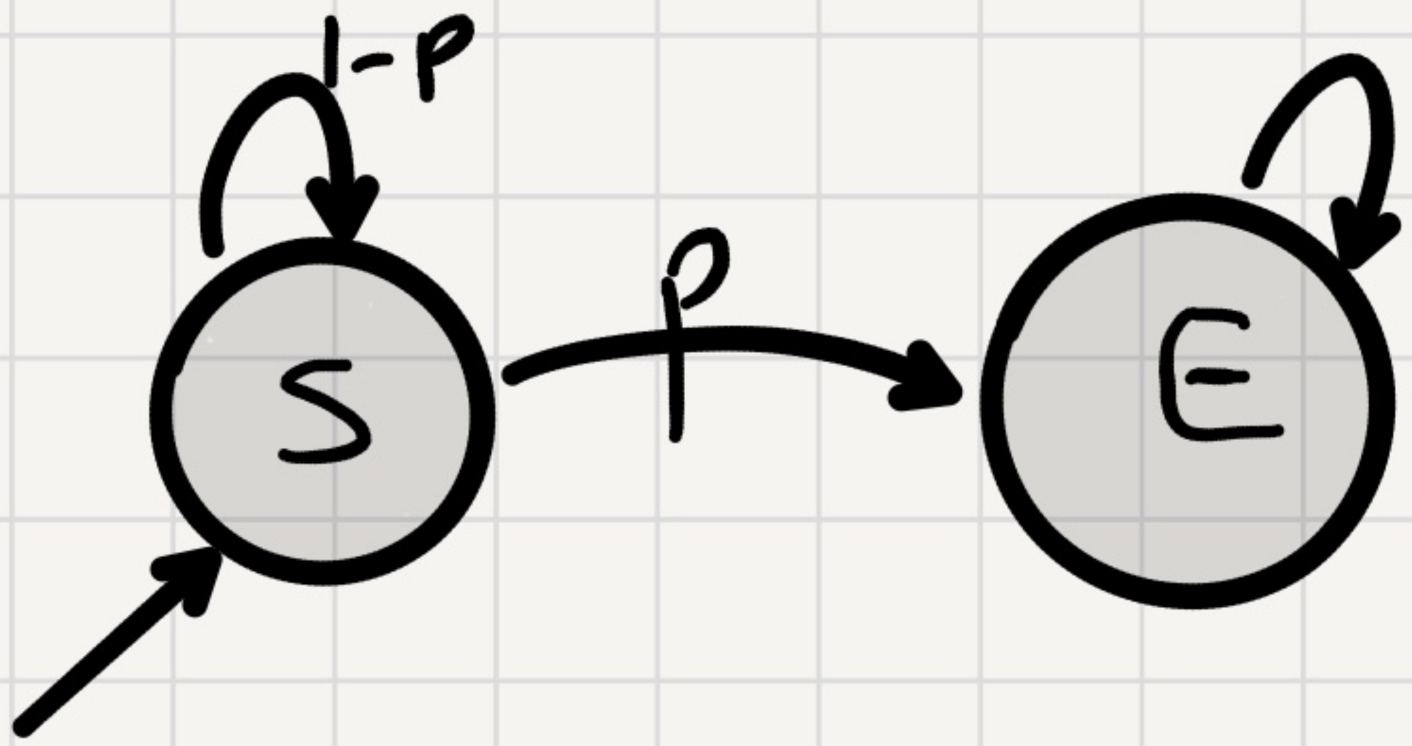
A bit more formal:

Let N be the r.v. capturing the number of steps until we hit E.

$$\beta(S) = \mathbb{E}[N] = \Pr[H]\cdot\mathbb{E}[N|H] + \Pr[T]\cdot\mathbb{E}[N|T]$$
$$= p\cdot 1 + (1-p)\cdot(1+\beta(S)) = 1 + (1-p)\beta(S).$$

Hitting Time - Example



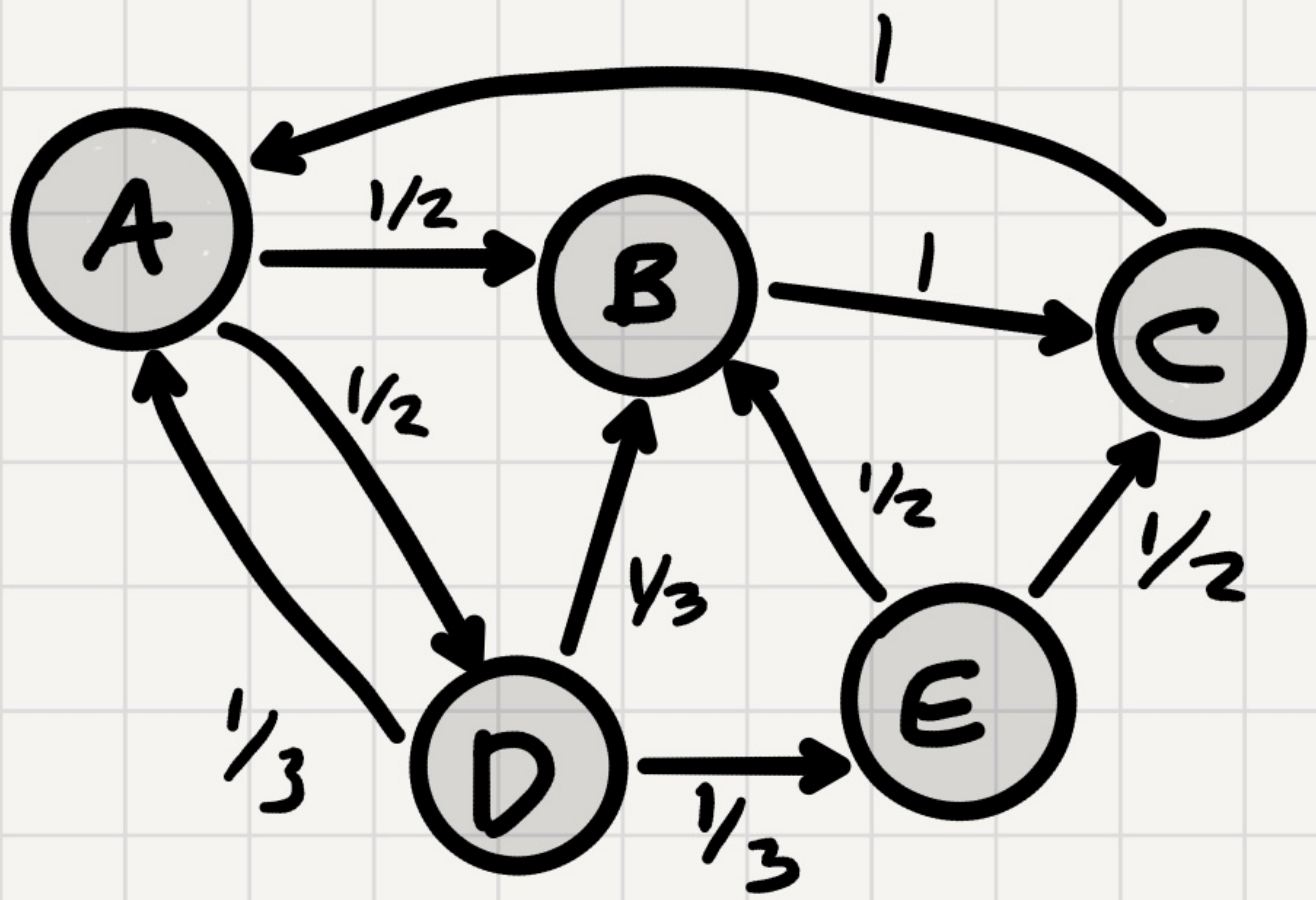
Let $\beta(S)$ = average time until a MC starting from S would reach E.

$$\boxed{\beta(S) = 1 + (1-p)\cdot\beta(S) + p\cdot 0.}$$

Solving: $p\cdot\beta(S) = 1 \implies \beta(S) = 1/p.$

We calculated the mean of a geometric R.V. without any infinite sums...

Hitting Time - Example 2



for $i \in \{A, B, C, D, E\}$ $\quad \beta(i)$ = average time until a MC starting at i would reach E.

$\beta(E) =$

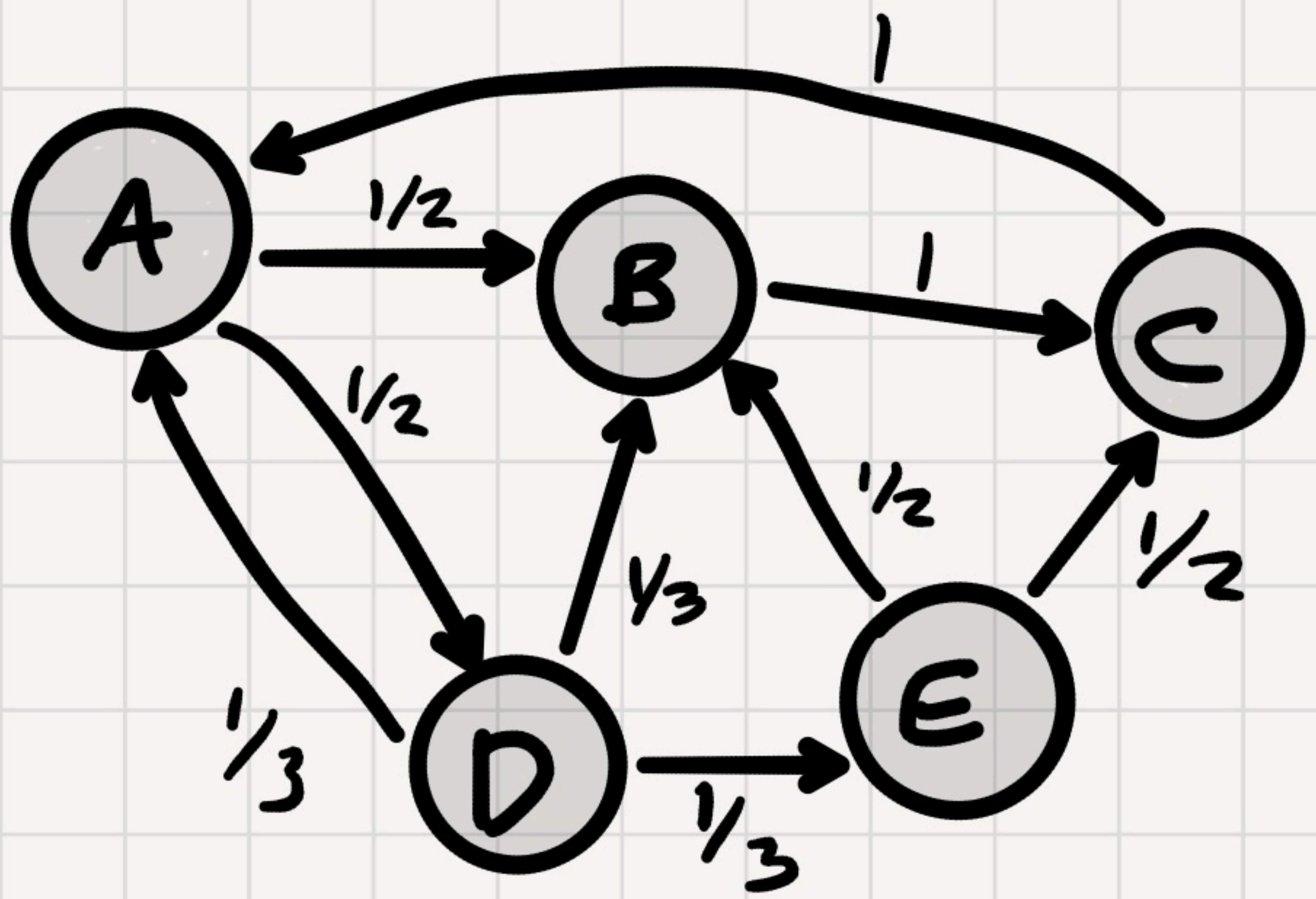
$\beta(A) =$

$\beta(B) =$

$\beta(C) =$

$\beta(D) =$

Hitting Time - Example 2



for $i \in \{A, B, C, D, E\}$ $\quad \beta(i) =$ average time until a MC starting at i would reach E.

$$\beta(E) = 0$$

$$\beta(A) = 1 + \frac{1}{2}\beta(B) + \beta(D)$$

$$\beta(B) = 1 + \beta(C)$$

$$\beta(C) = 1 + \beta(A)$$

$$\beta(D) = 1 + \frac{1}{3}\beta(A) + \frac{1}{3}\beta(B) + \frac{1}{3}\beta(E)$$

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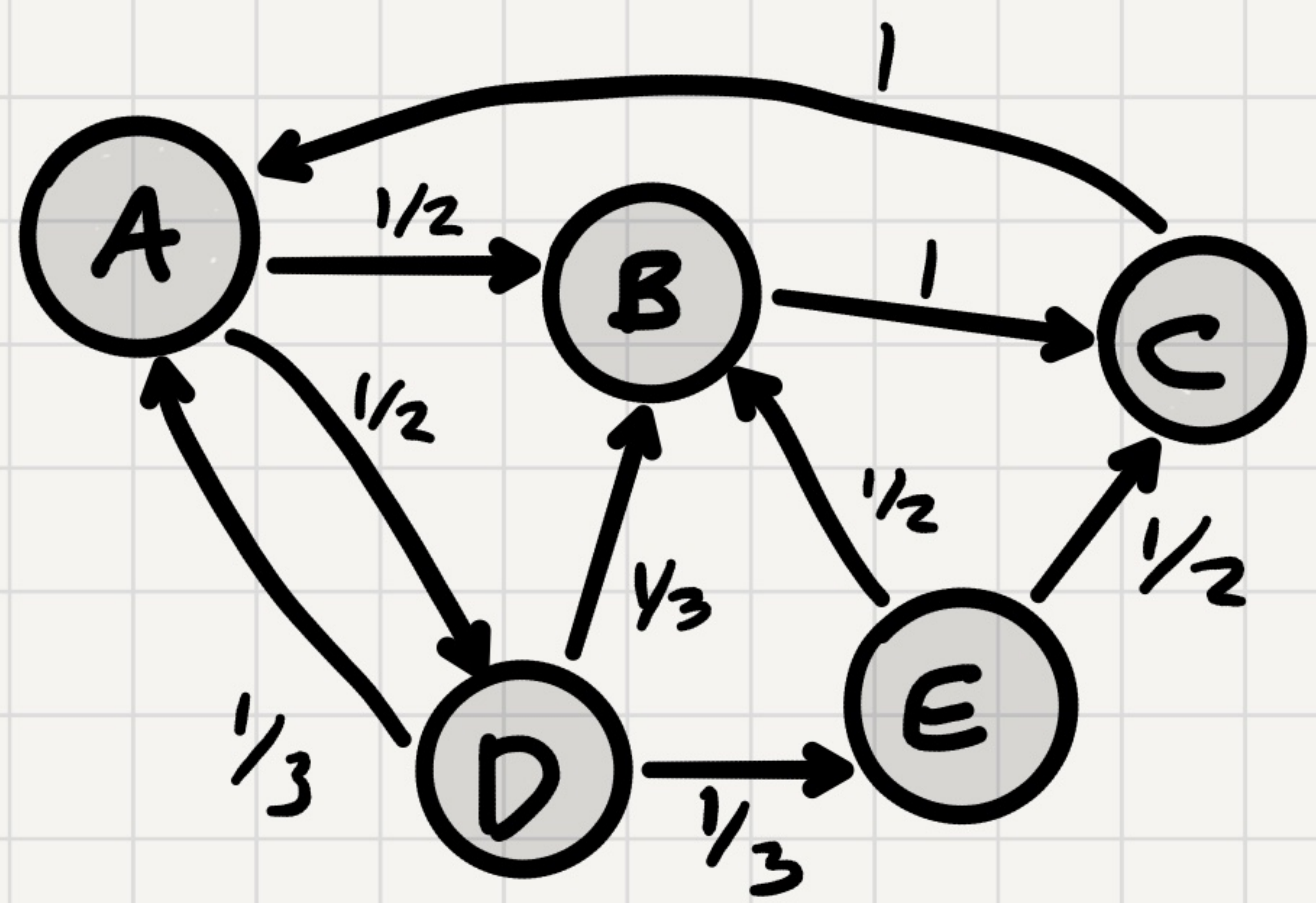

for $i \in \{A, B, C, D, E\}$ $\beta(i)$ = average time until a MC starting at i would reach E.

$$\left.\begin{aligned}
\beta(E) &= 0 \\
\beta(A) &= 1 + \tfrac{1}{2}\beta(B) + \tfrac{1}{2}\beta(D) \\
\beta(B) &= 1 + \beta(C) \\
\beta(C) &= 1 + \beta(A) \\
\beta(D) &= 1 + \tfrac{1}{3}\beta(A) + \tfrac{1}{3}\beta(B)
\end{aligned}\right\}$$

Solve the system of linear equations:

$\beta(A) = 17$
$\beta(B) = 19$
$\beta(C) = 18$
$\beta(D) = 13$

Hitting Time - Example 3

You flip a coin w. heads prob. p until you get two consecutive H.

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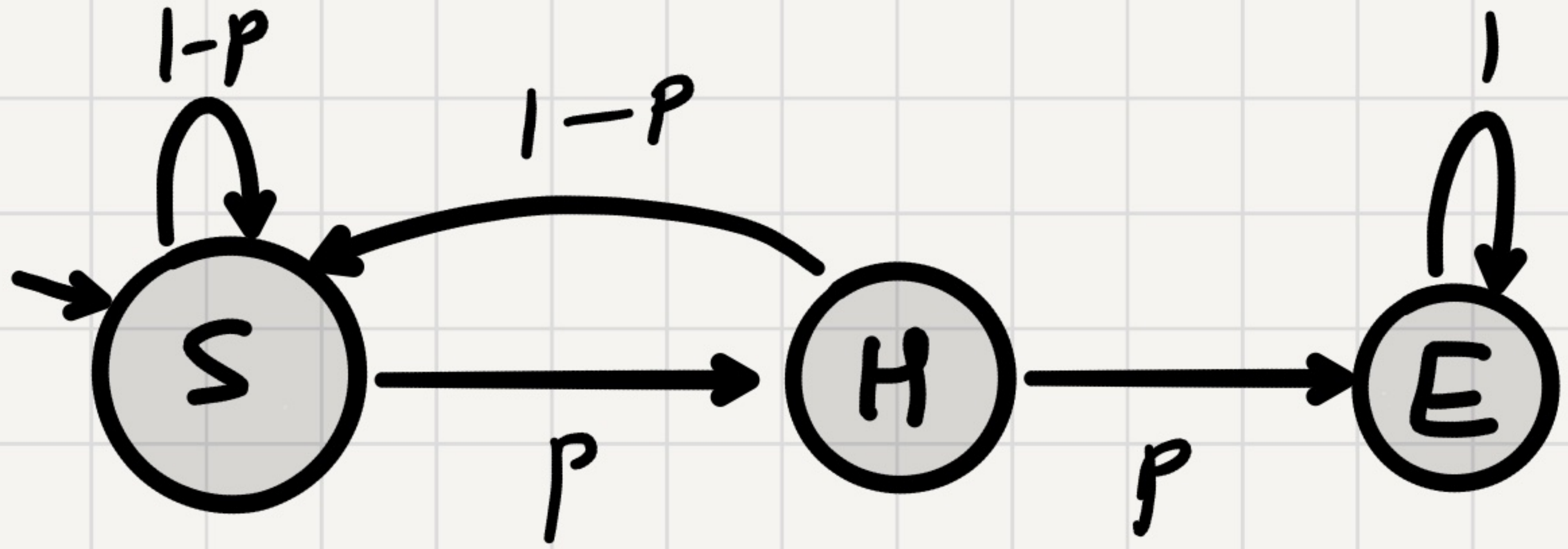

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You flip a coin w. heads prob. p until you get two consecutive H.

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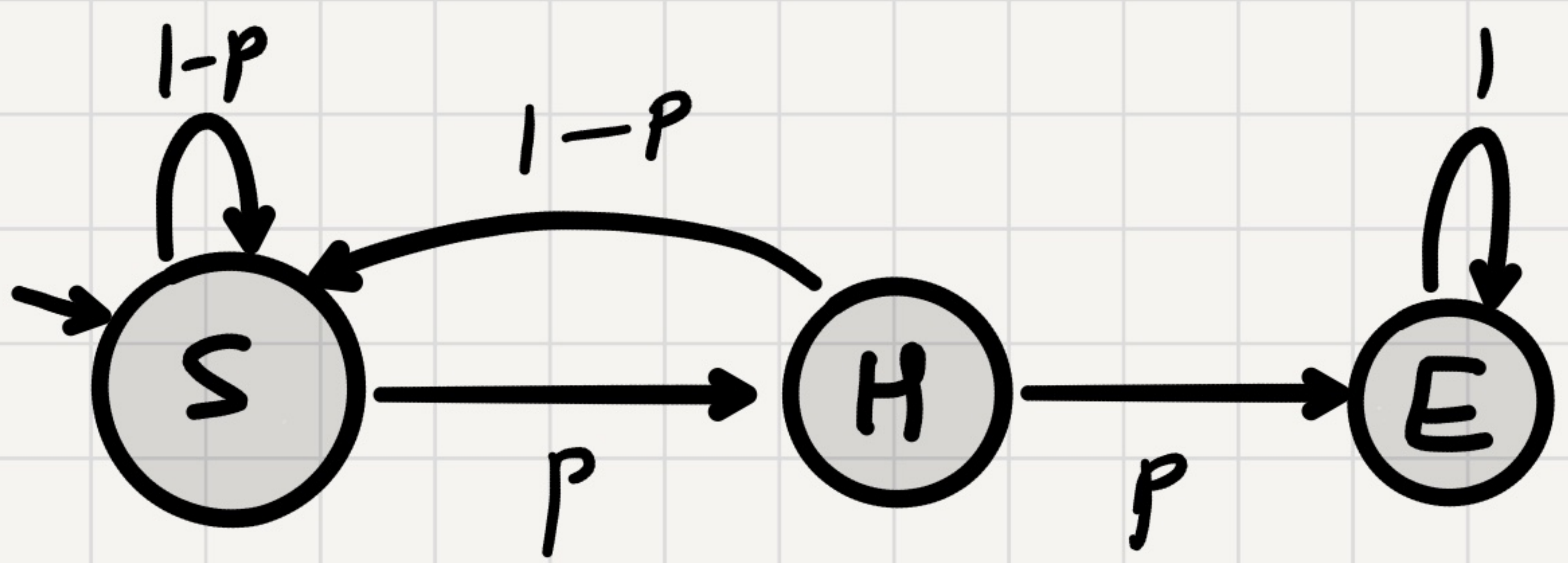


$\beta(E) = 0$

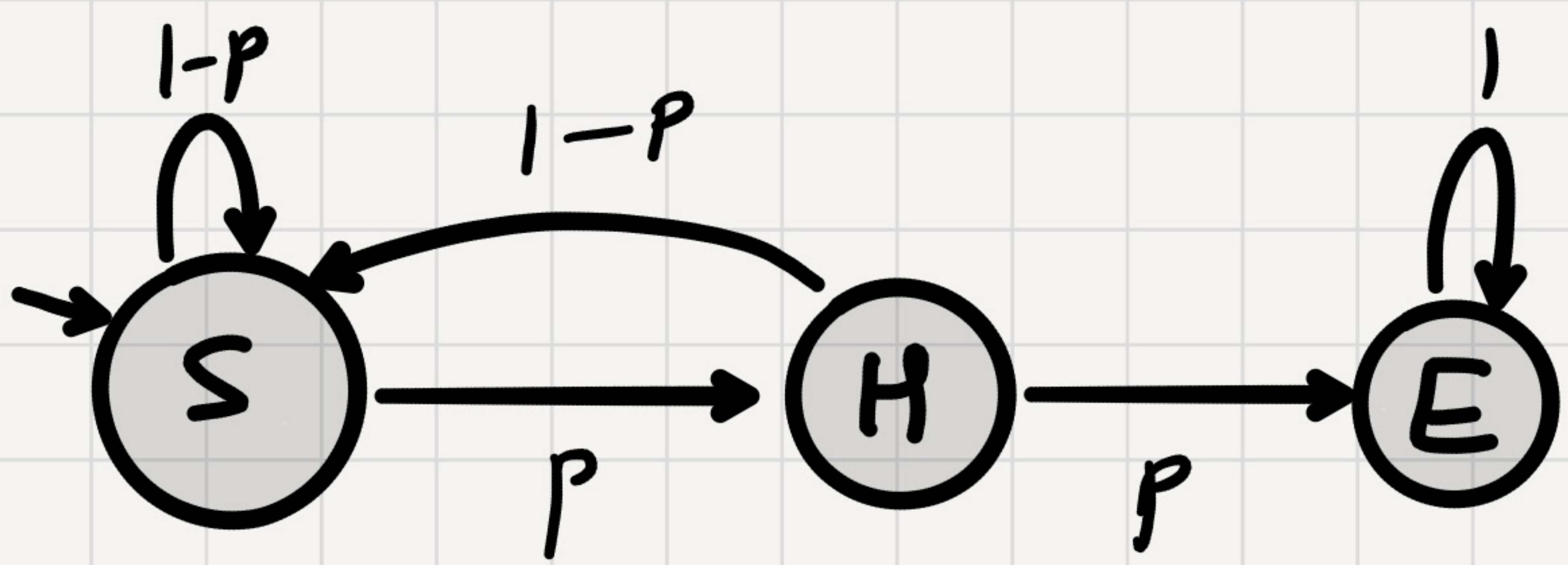
$\beta(H) = 1 + (1-p) \cdot \beta(S) + p \cdot \beta(E)$

$\beta(S) = 1 + (1-p)\beta(S) + p \cdot \beta(H)$

Hitting Time - Example 3

You flip a coin w. heads prob. p until you get two consecutive H.

How many flips on average? $\boxed{\frac{1+p}{p^2}}$



$\beta(E) = 0$

$\beta(H) = 1 + (1-p)\cdot\beta(S) + \cancel{p\cdot\beta(E)}$

$\beta(S) = 1 + (1-p)\beta(S) + p\cdot\beta(H)$

$= 1 + (1-p)\beta(S) + p\cdot(1+(1-p)\beta(S))$

$\therefore \; \beta(S) = \frac{1+p}{p^2}$.

Here Before There

You go to a nice casino. You start with 10 \$

Every round, you win 1 \$ w.p. 50%

lose 1 \$ w.p. 50%

What's the probability you'll reach 100 \$ before 0 \$?

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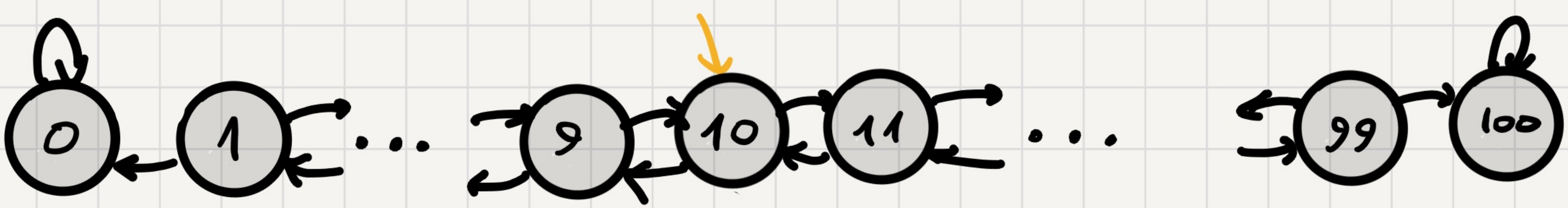

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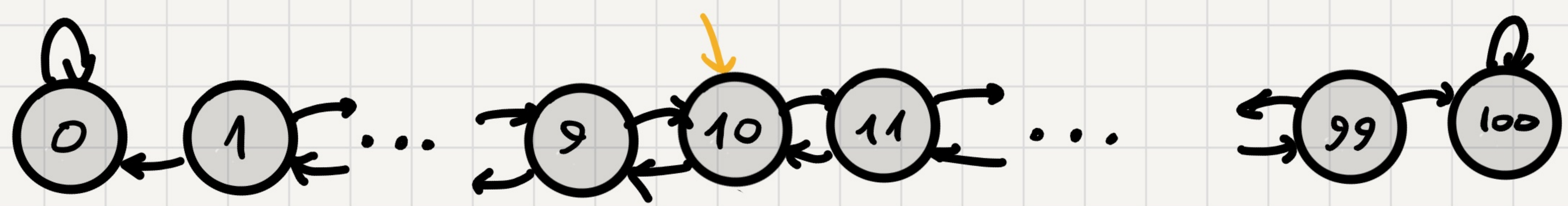

For $i \in \{0, 1, \dots, 100\}$

let $\alpha(i)$ = Prob. of reaching 100 before 0 starting from i.

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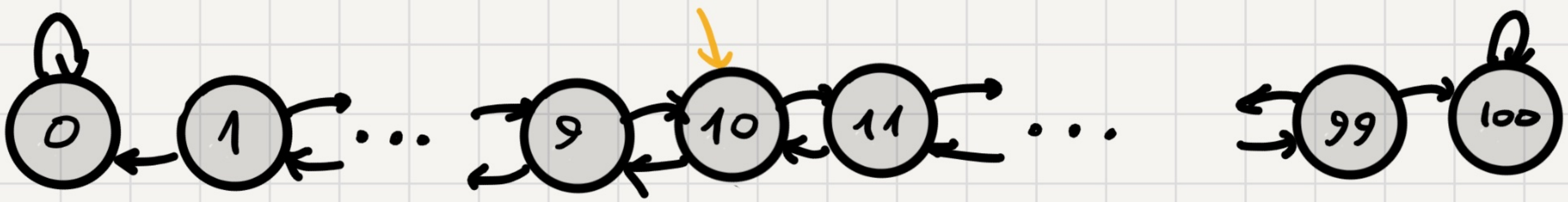

For $i = 0, 1, \ldots, 100$,

$\alpha(i)$ = Prob. of reaching 100 before 0 starting from i.

What's true:

A. $\alpha(0) = 0$

B. $\alpha(0) = 1$

C. $\alpha(100) = 1$

D. $\alpha(i) = 1 + \frac{1}{2}\alpha(i-1) + \frac{1}{2}\alpha(i+1)$ for $1 \leq i \leq 99$.

E. $\alpha(i) = \frac{1}{2}\alpha(i-1) + \frac{1}{2}\alpha(i+1)$ for $1 \leq i \leq 99$.

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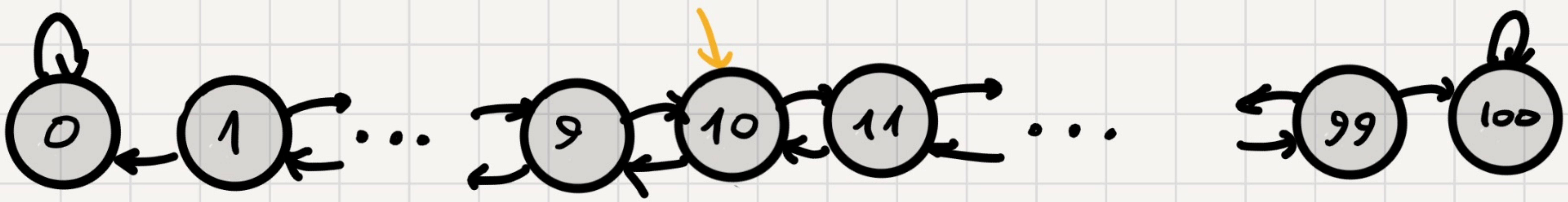


For $i = 0, 1, \ldots, 100$,

$\alpha(i)$ = Prob. of reaching 100 before 0 starting from i.

$\alpha(0) = 0 \qquad \alpha(100) = 1$

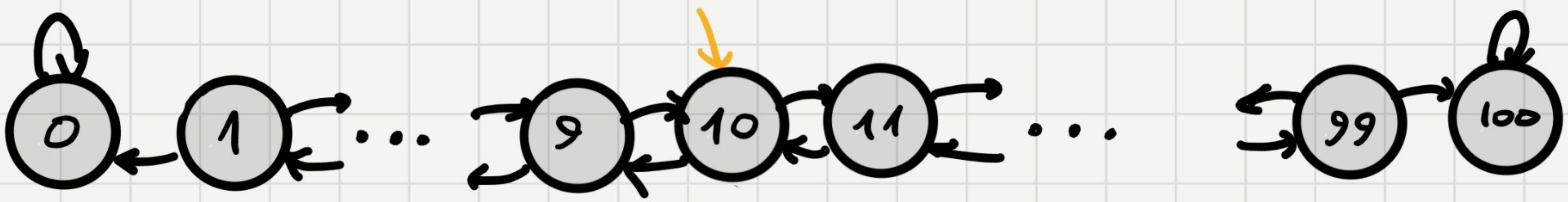
For $i = 1, \ldots, 99 \qquad \alpha(i) = \frac{1}{2}\alpha(i-1) + \frac{1}{2}\alpha(i+1)$

$\alpha(i)$ = avg of $\alpha(i-1), \alpha(i+1)$.

What's the solution?

Here Before There

What's the probability you'll reach 100\$ before 0\$?



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$\alpha(i) =$ avg of $\alpha(i-1), \alpha(i+1)$.

What's the solution?

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You go to a real casino. You start with 10 \$

Every round, you win 1 \$ w.p. 48 %

lose 1 \$ w.p. 52 %

What's the probability you'll reach 100 \$ before 0 \$?

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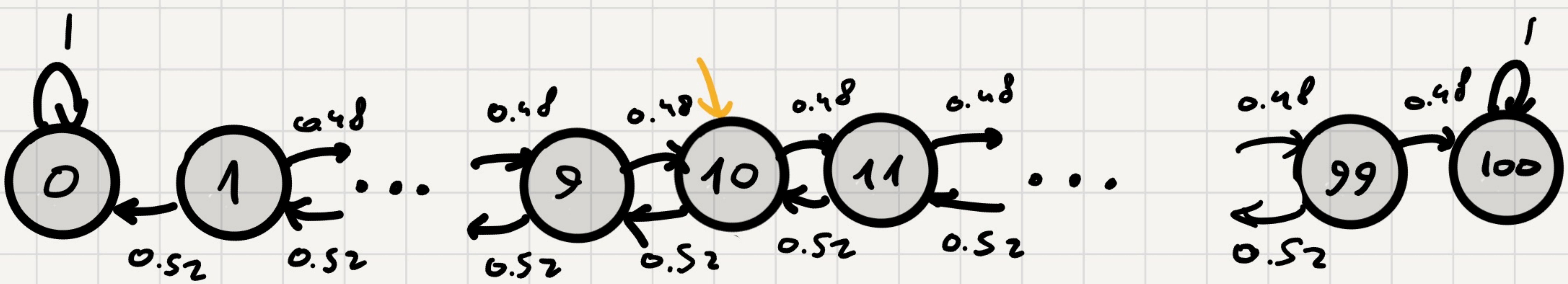

$\alpha(0) = 0 \qquad \alpha(100) = 1$

$$\alpha(i) = 0.48 \cdot \alpha(i+1) + 0.52 \cdot \alpha(i-1)$$

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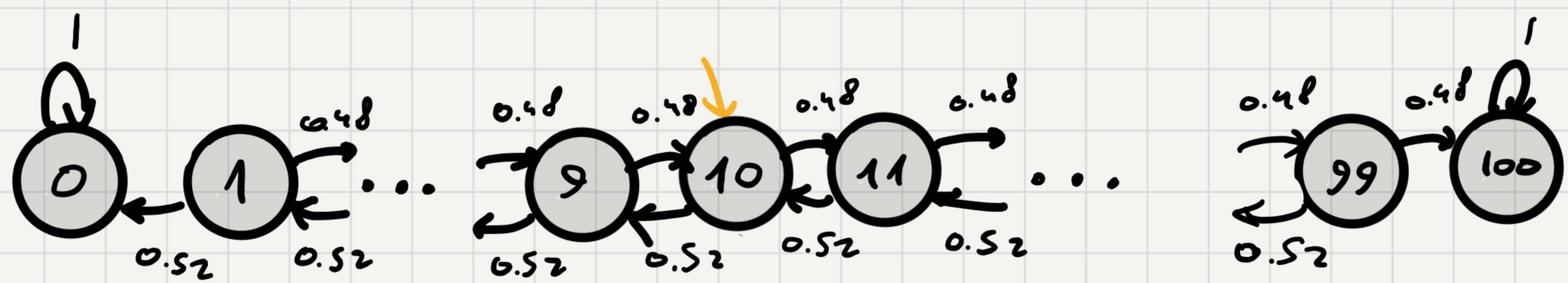
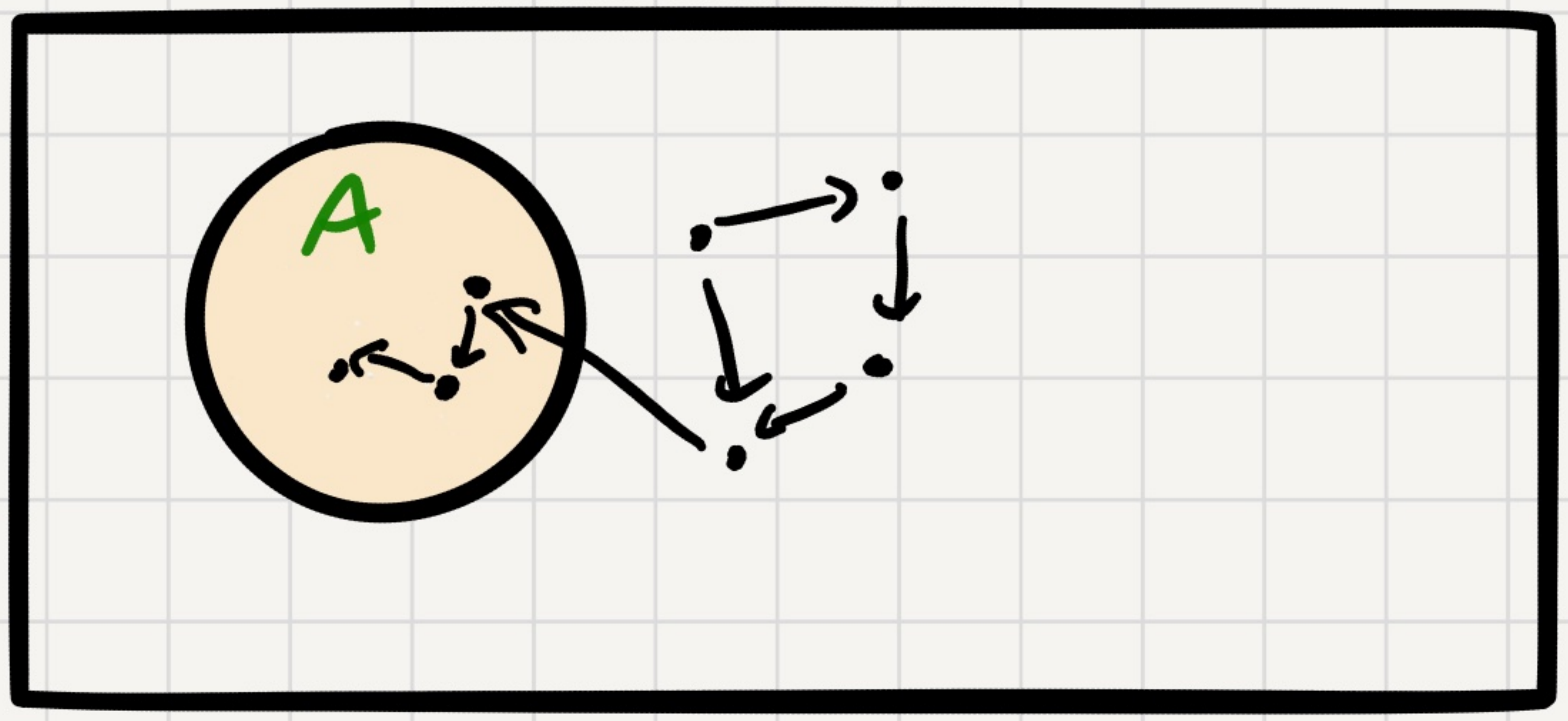


$\alpha(0)=0 \qquad \alpha(100)=1$

$$\alpha(i) = 0.48 \cdot \alpha(i+1) + 0.52 \cdot \alpha(i-1)$$

Solution: $\alpha(i) = \dfrac{\rho^i - 1}{\rho^{100} - 1}$ where $\rho = \dfrac{0.52}{0.48}$. $\alpha(10) \approx \dfrac{1}{2{,}440}$

First Step Equations



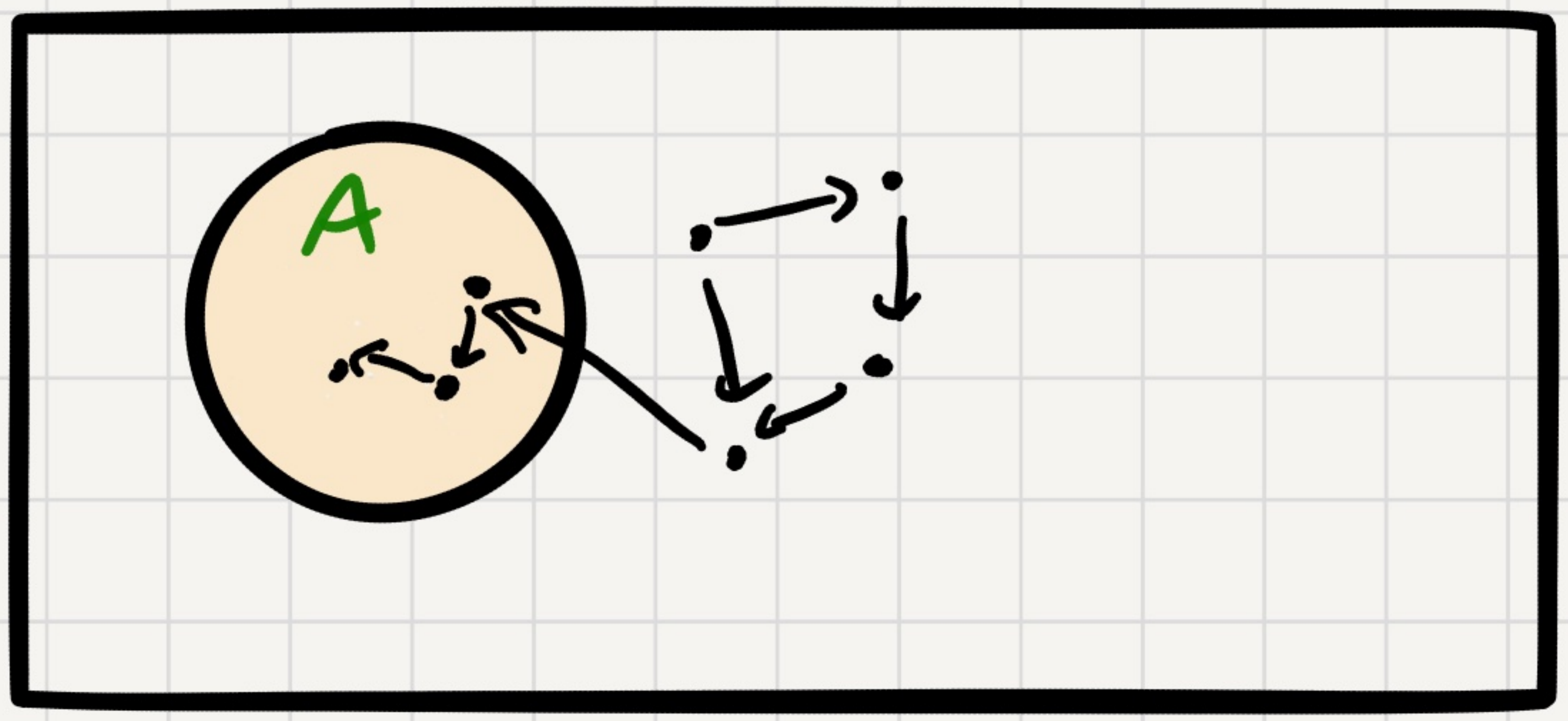
Let $\{X_n\}_{n=0}^{\infty}$ be a MC on $\mathcal{X}$ $A \subseteq \mathcal{X}$

$\beta(i)$ = expected time to reach A starting from i.

Formally: Define the r.v. $T_A = \min\{n \geq 0 : X_n \in A\}$

$$\beta(i) = \mathbb{E}[T_A \mid X_0 = i]$$

First Step Equations



Let $\{X_n\}_{n=0}^{\infty}$ be a MC on $\mathcal{X}$ $\quad A \subseteq \mathcal{X}$

$\beta(i)$ = expected time to reach A starting from i.

$\beta(i) = \boxed{}$ for $i \in A$

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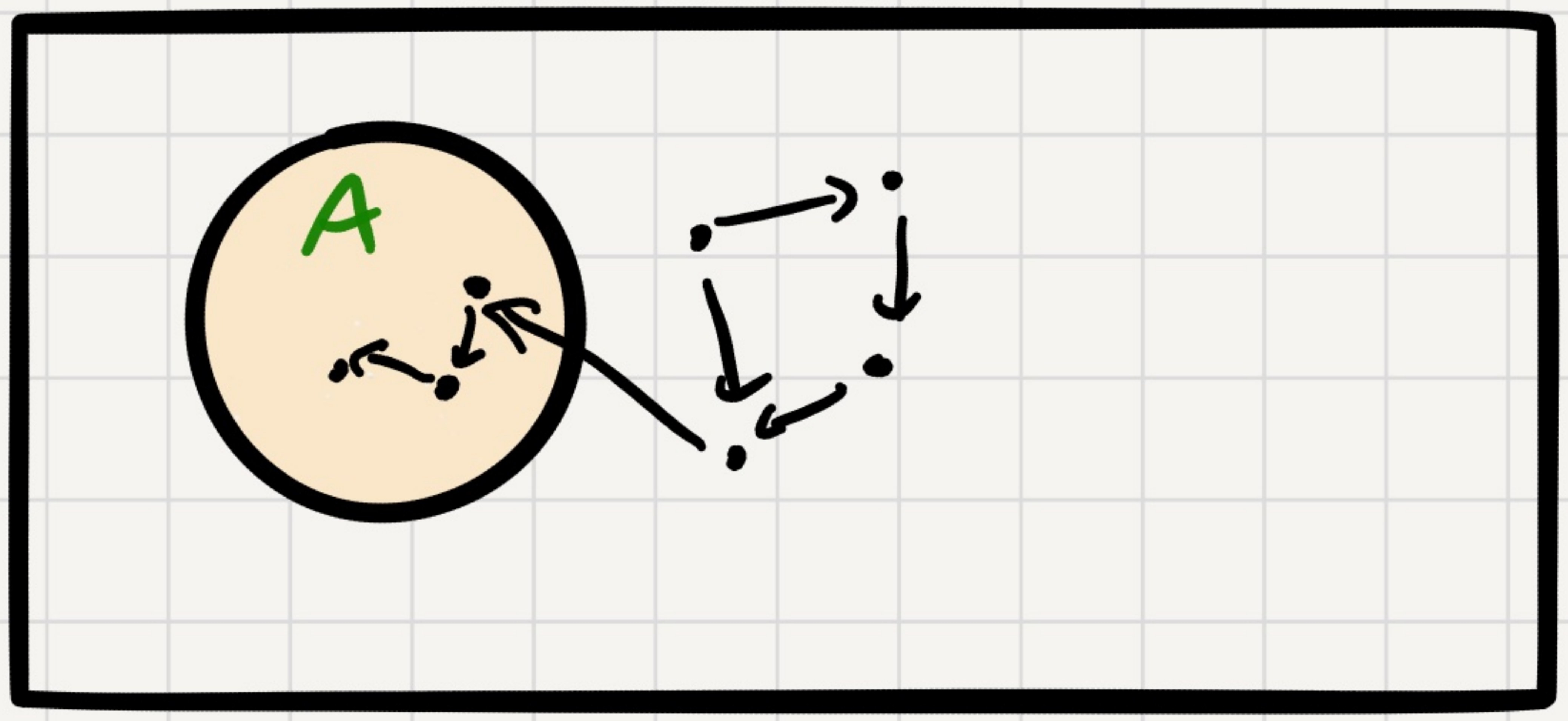
First Step Equations



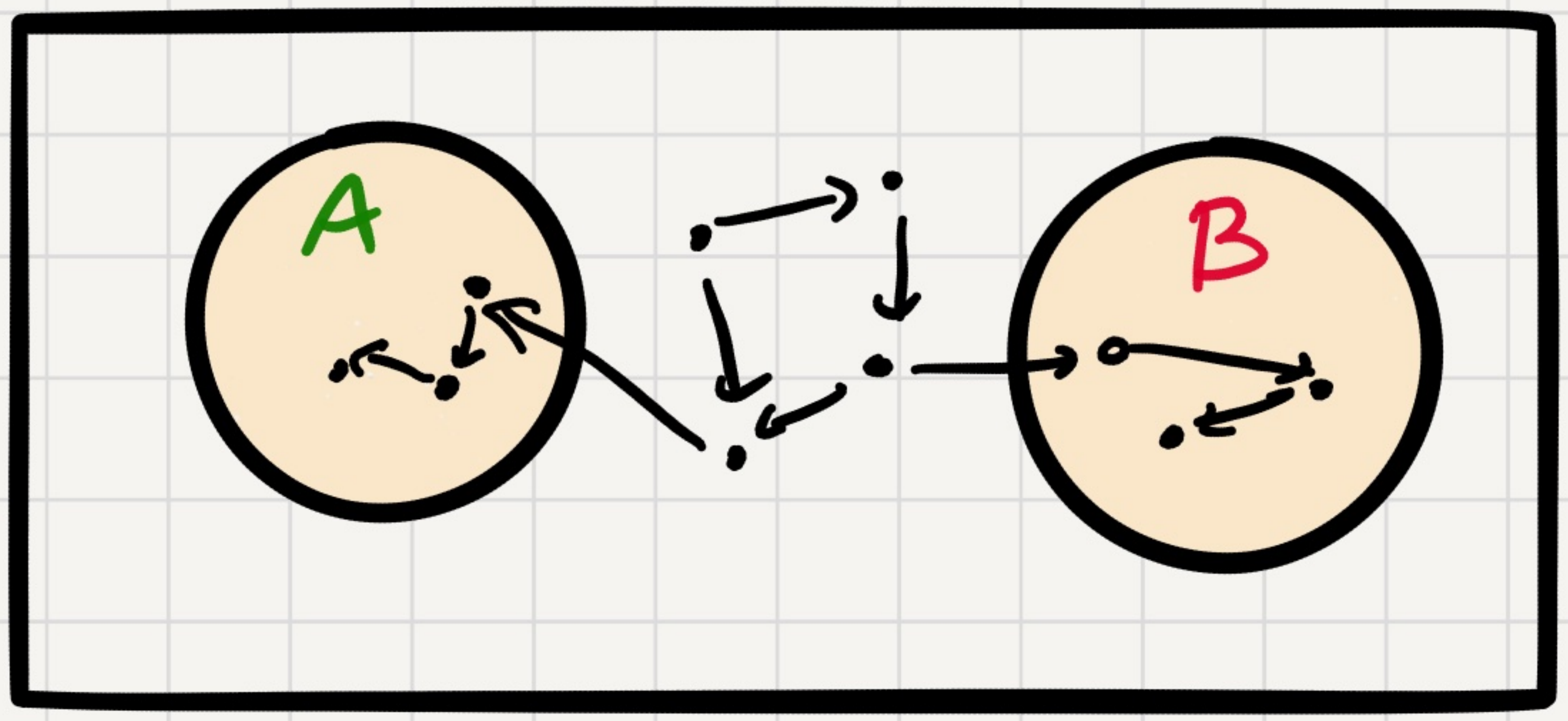
Let $\{X_n\}_{n=0}^{\infty}$ be a MC on $\mathcal{X}$ $\quad A \subseteq \mathcal{X}$

$\beta(i)$ = expected time to reach A starting from i.

$\beta(i) = 0 \quad$ for $i \in A$

$\beta(i) = 1 + \sum_j P(i,j)\,\beta(j) \quad$ for $i \notin A$.

First Step Equations



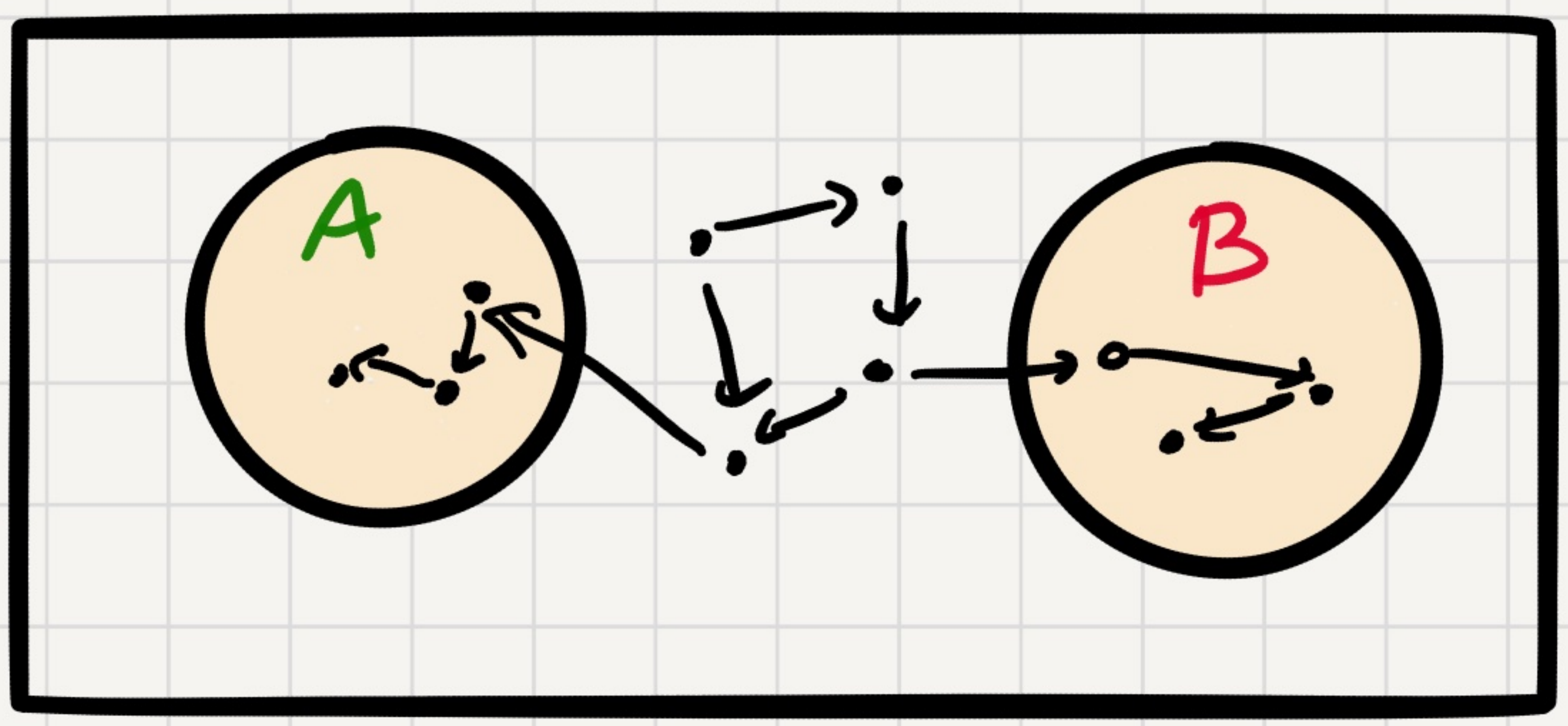
Let $\{X_n\}_{n=0}^{\infty}$ be a MC on $\mathcal{X}$ $\quad A, B \subseteq \mathcal{X}$

A & B disjoint.

$\alpha(i) = \Pr[\text{reaching } A \text{ before } B, \text{starting from } i]$

$$\begin{cases} \alpha(i) = \boxed{\qquad\qquad} & \text{for } i \in B \\ \alpha(i) = \boxed{\qquad\qquad} & \text{for } i \in A \\ \alpha(i) = \boxed{\qquad\qquad\qquad\qquad\qquad\qquad} & \text{for } i \notin A \cup B. \end{cases}$$

First Step Equations



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$$\begin{cases} \alpha(i) = 0 & \text{for } i \in B \\ \alpha(i) = 1 & \text{for } i \in A \\ \alpha(i) = \sum_j P(i,j) \cdot \alpha(j). & \text{for } i \notin A \cup B. \end{cases}$$

Distribution of X_n

Recall if a MC starts with distribution π_0

and has transition matrix P

then the dist of X_1 is $\pi_0 \cdot P$.

and $\forall n$ the dist of X_n is $\pi_0 \cdot P^n$.

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Suppose π_0 is a stationary dist.

What's π_1 ?

What's π_n ?

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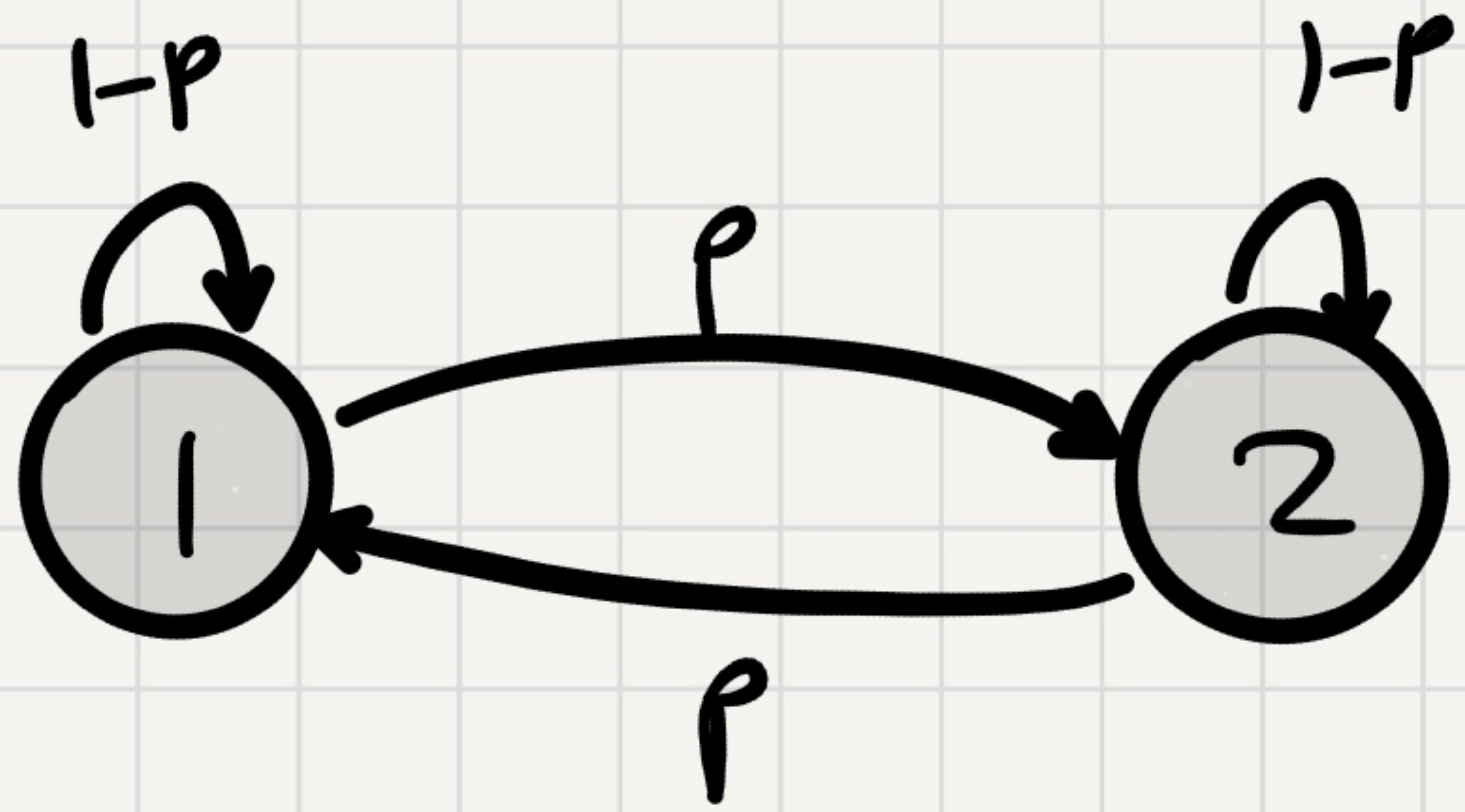
A distribution π is stationary if $\pi = \pi \cdot P$.

Theorem: π_0 is stationary iff $\forall n \geq 0$: $\pi_n = \pi_0$.

Stationary Distribution - Example



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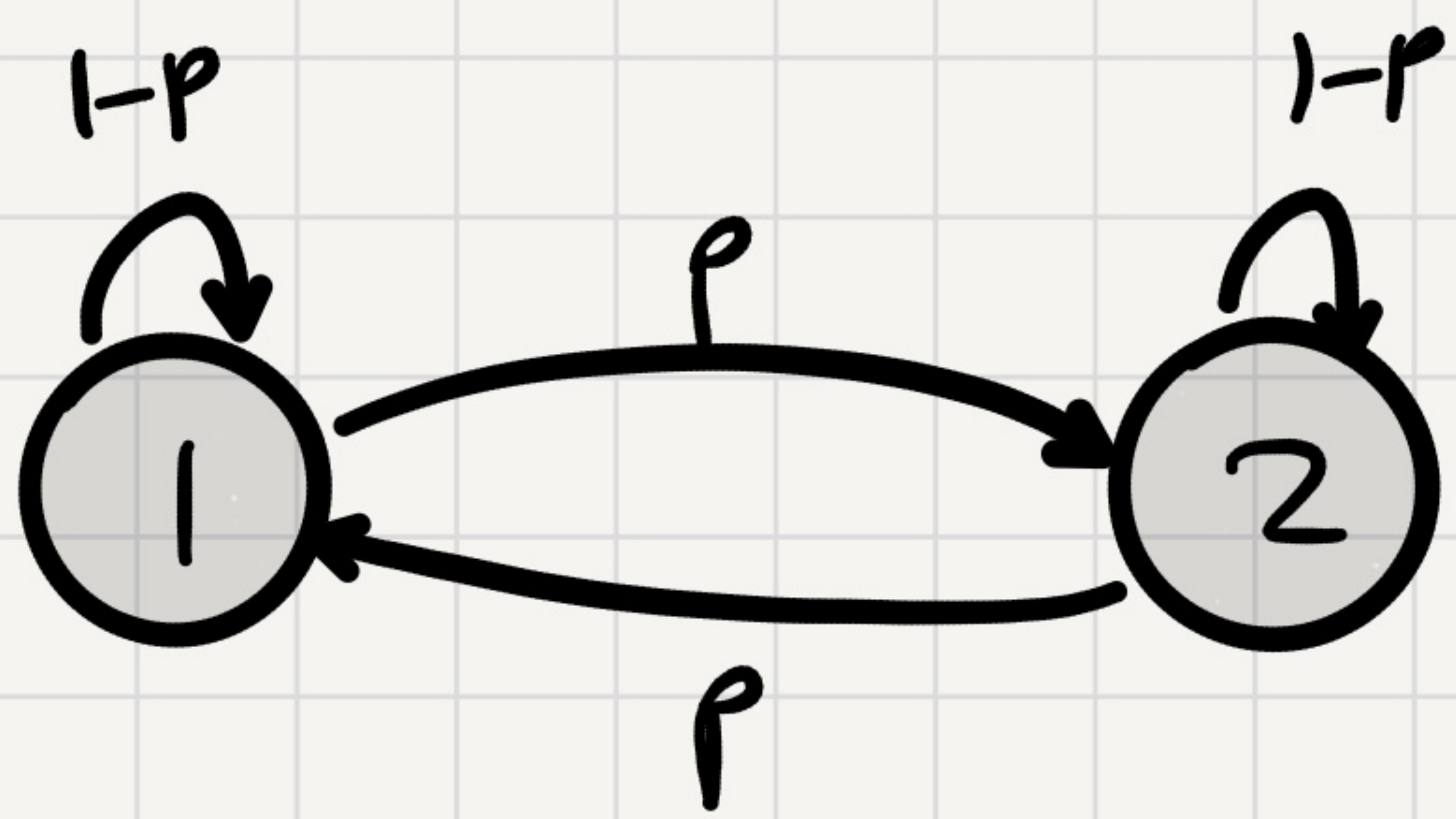


π is stationary iff $\begin{pmatrix}\pi(1) & \pi(2)\end{pmatrix} = \begin{pmatrix}\pi(1) & \pi(2)\end{pmatrix} \cdot \begin{pmatrix} 1-p & p \\ p & 1-p \end{pmatrix}$

$$\pi(1) = \pi(1)\cdot(1-p) + \pi(2)\cdot p \iff \pi(1) = \pi(2)$$

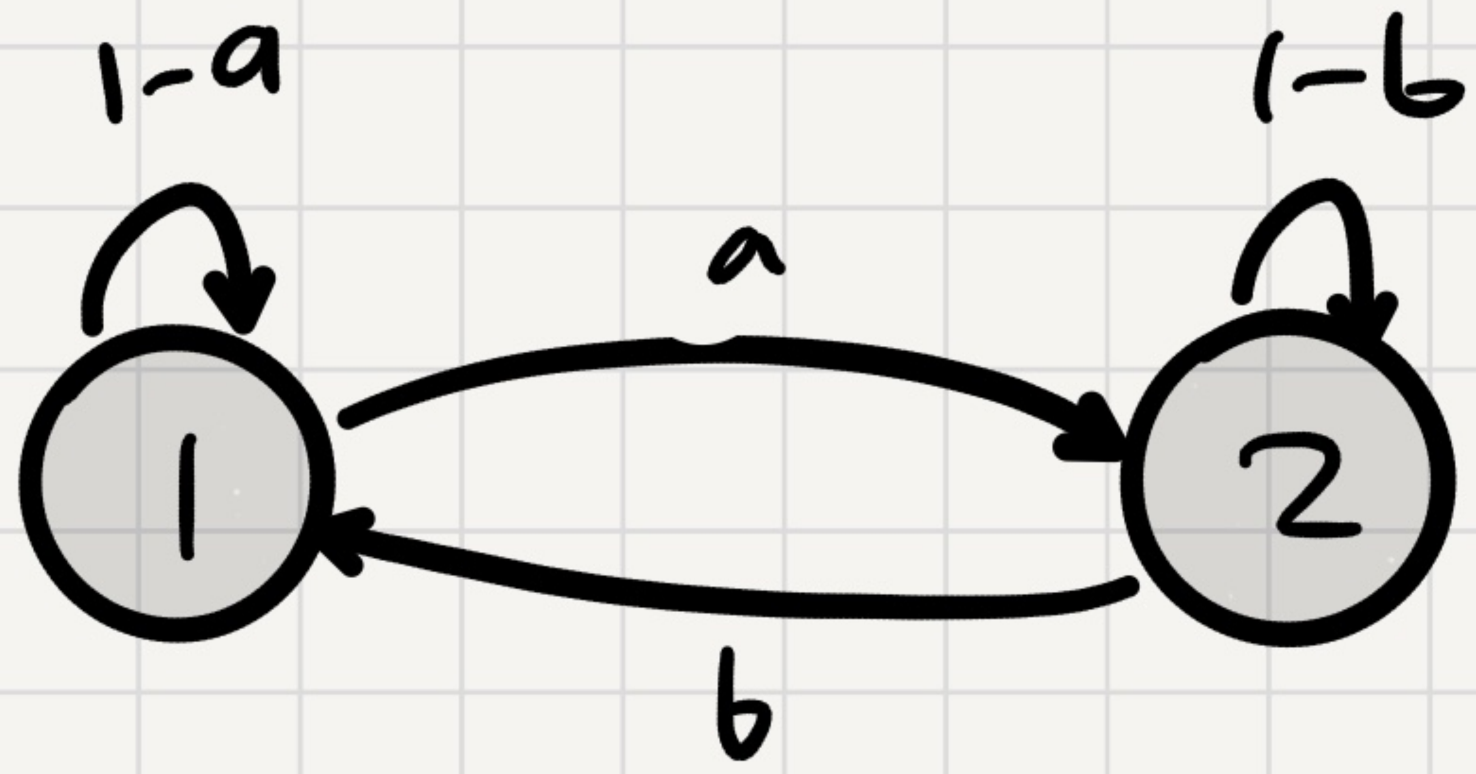
$$\pi(2) = \pi(1)\cdot p + \pi(2)\cdot(1-p) \iff \pi(1) = \pi(2)$$

The two equations are redundant.

Is there a unique solution?

Yes, if we also use $\pi(1) + \pi(2) = 1$.

Stationary Distribution - Example 2



π is stationary iff $\begin{pmatrix}\pi(1) & \pi(2)\end{pmatrix} = \begin{pmatrix}\pi(1) & \pi(2)\end{pmatrix} \cdot \begin{pmatrix} 1-a & a \\ b & 1-b \end{pmatrix}$

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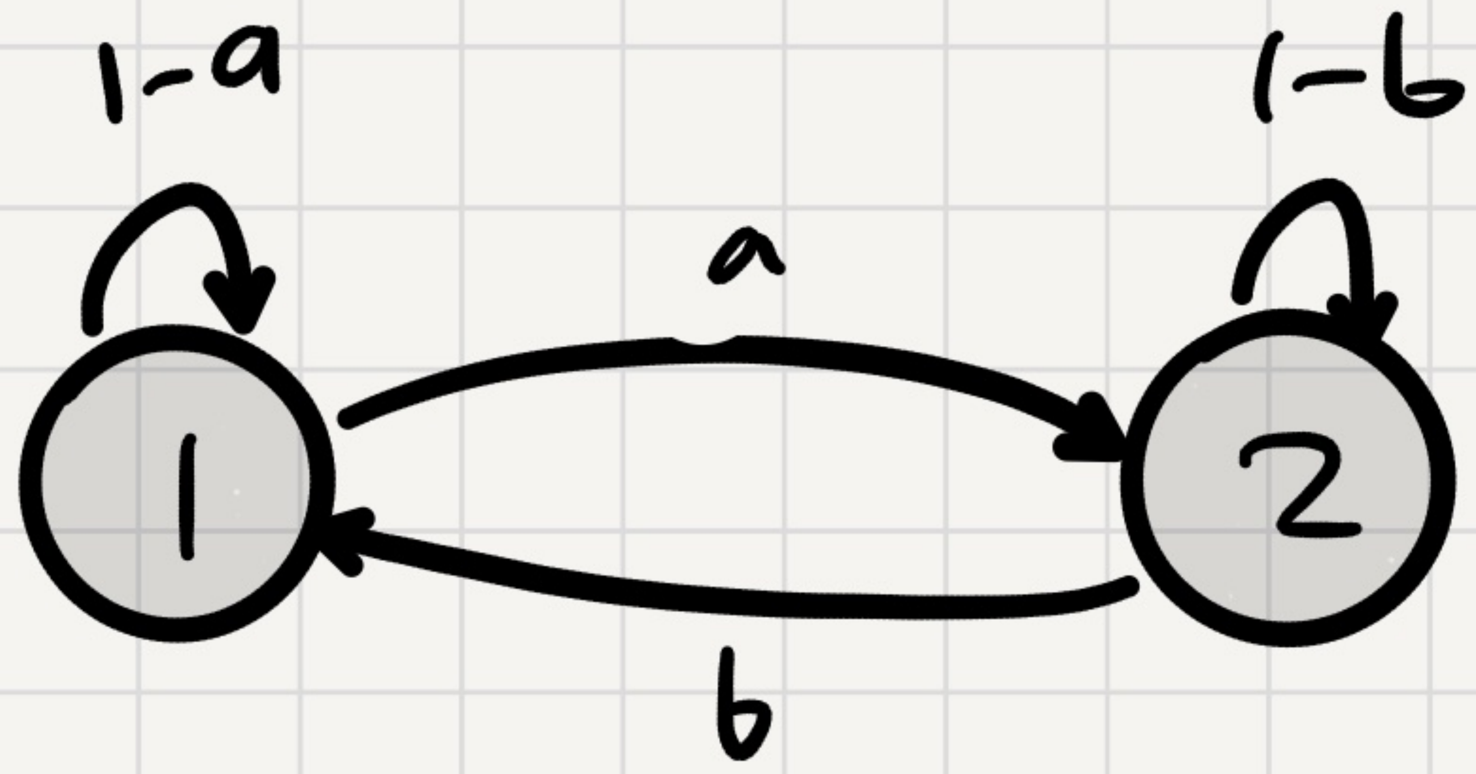


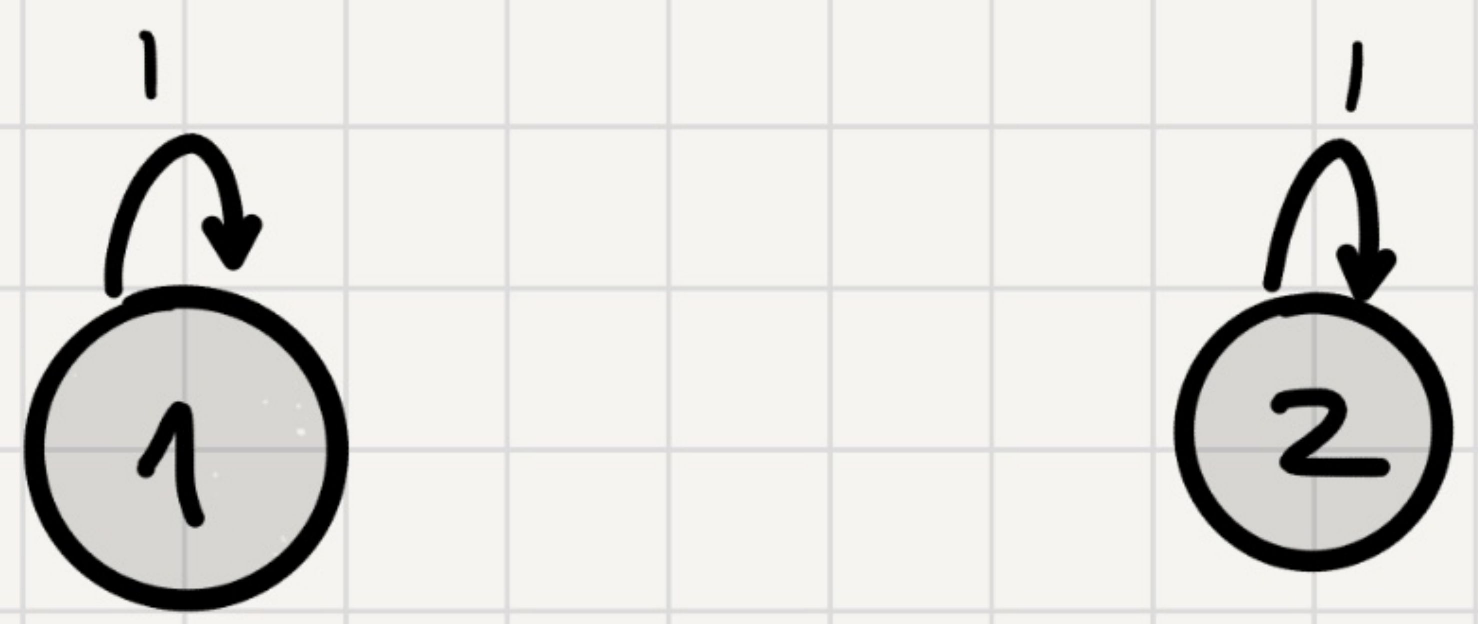
π is stationary iff $(\pi(1)\ \pi(2)) = (\pi(1)\ \pi(2)) \cdot \begin{pmatrix} 1-a & a \\ b & 1-b \end{pmatrix}$

and $\pi(1) + \pi(2) = 1$

$$\pi(1) = \pi(1) \cdot (1-a) + \pi(2) \cdot b \iff a \cdot \pi(1) = \pi(2) \cdot b$$

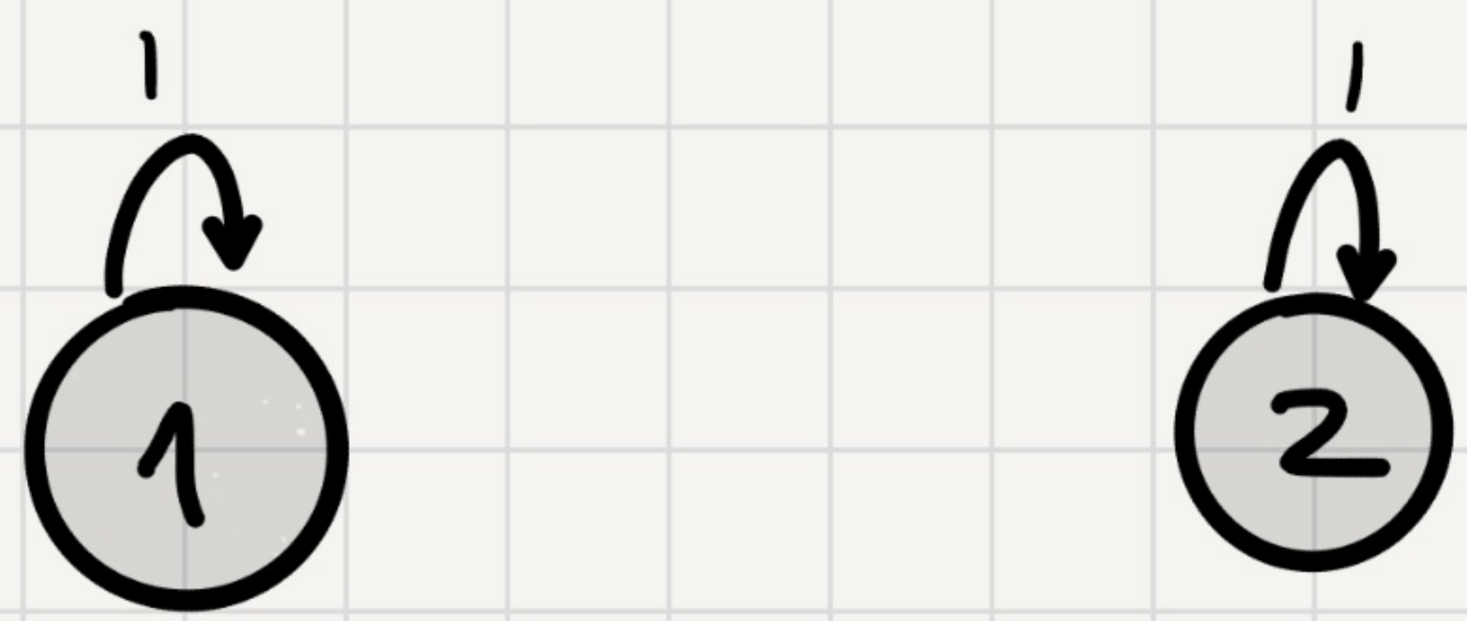
Unique Solution: $\pi = \left[\frac{b}{a+b}, \frac{a}{a+b} \right]$

Stationary Distributions - Example 3



Which distributions are stationary?

Stationary Distributions - Example 3

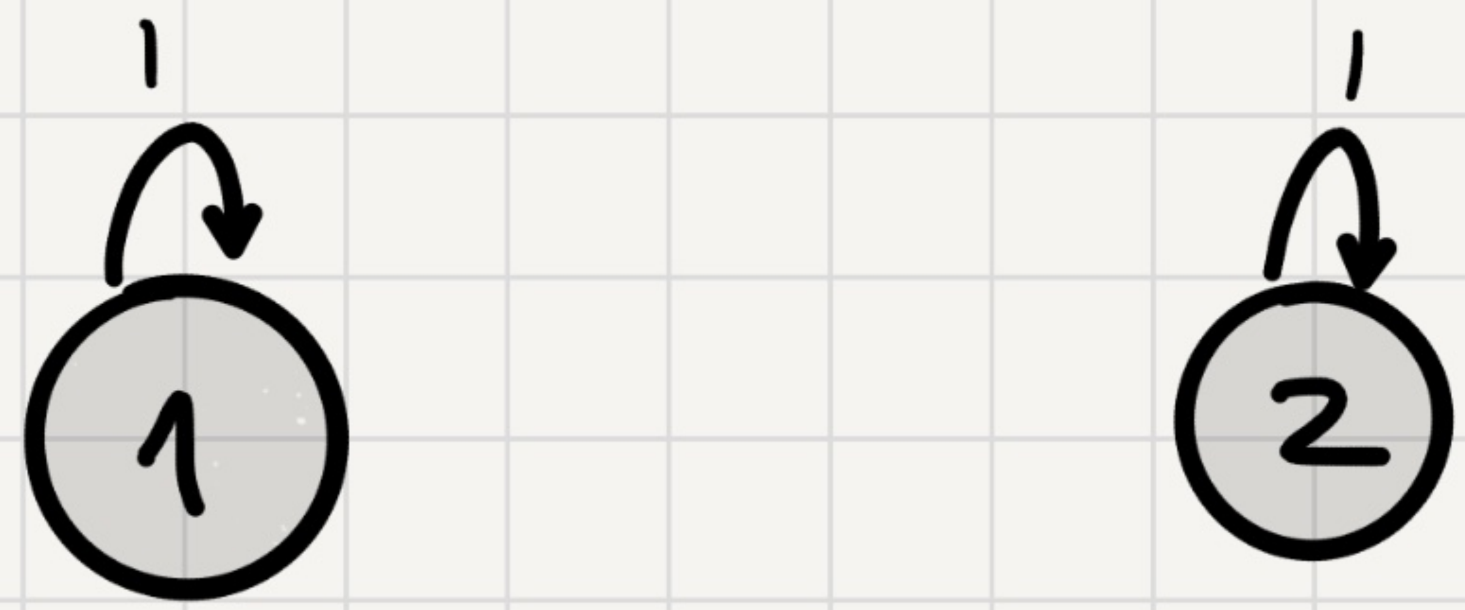


Which distributions are stationary?

all of them.

$$\forall \pi \qquad \pi = \pi \cdot \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Stationary Distributions - Example 3



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Q: Which Markov Chains have a unique stationary distribution?

Irreducible Markov Chains

A MC is irreducible if you can go from every state i to every state j (possibly in multiple steps).

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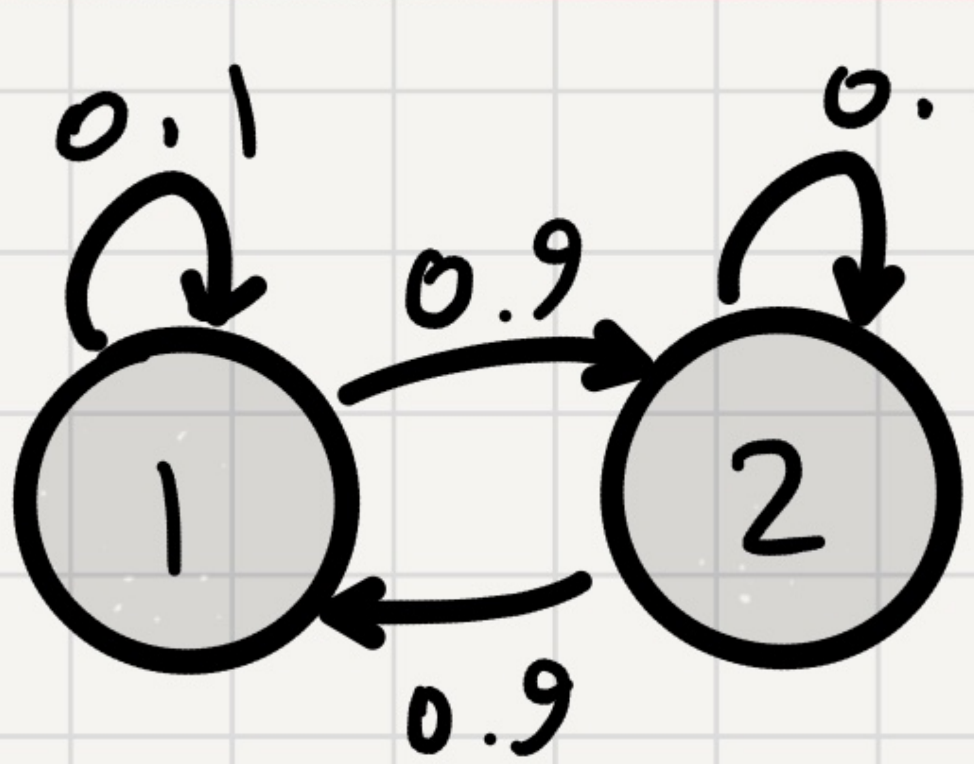
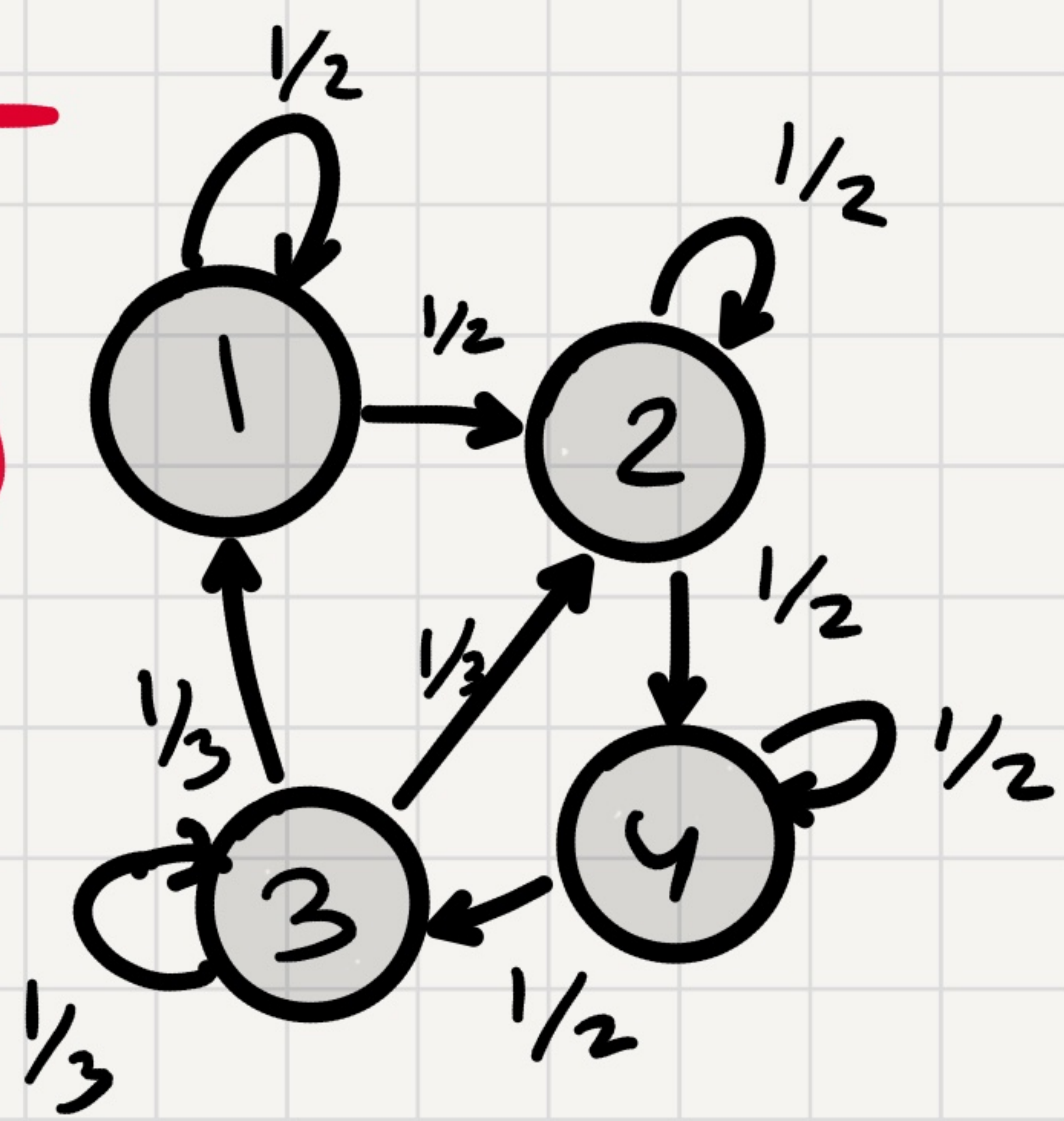
Which MC are irreducible?

(A)



(B)



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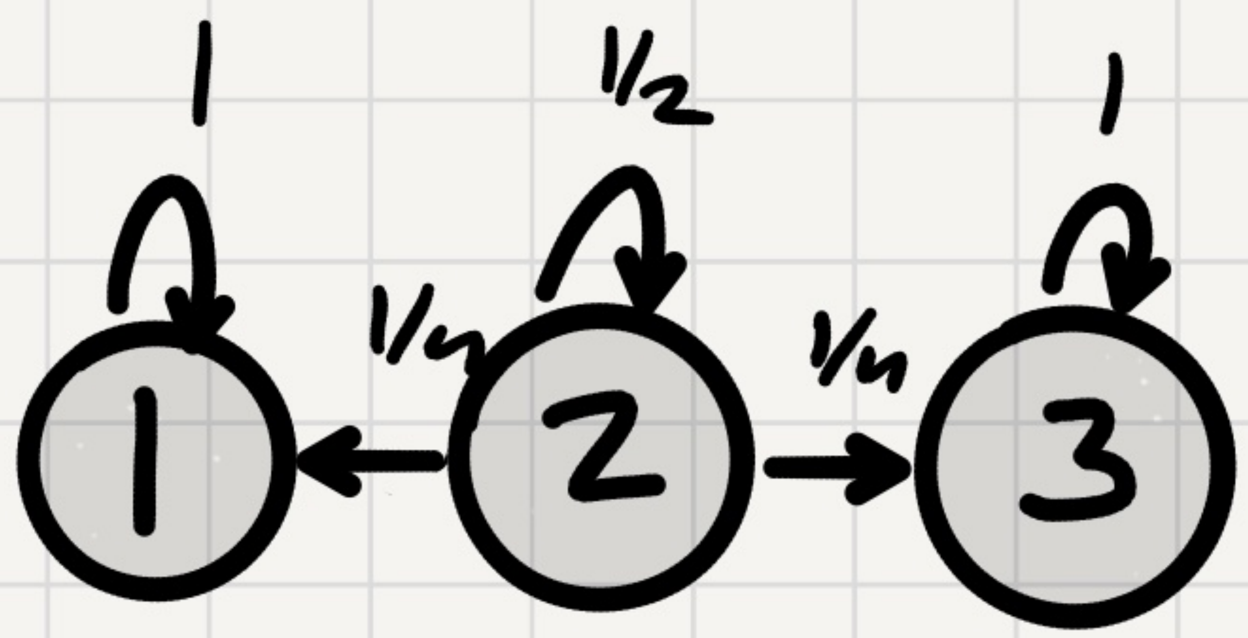
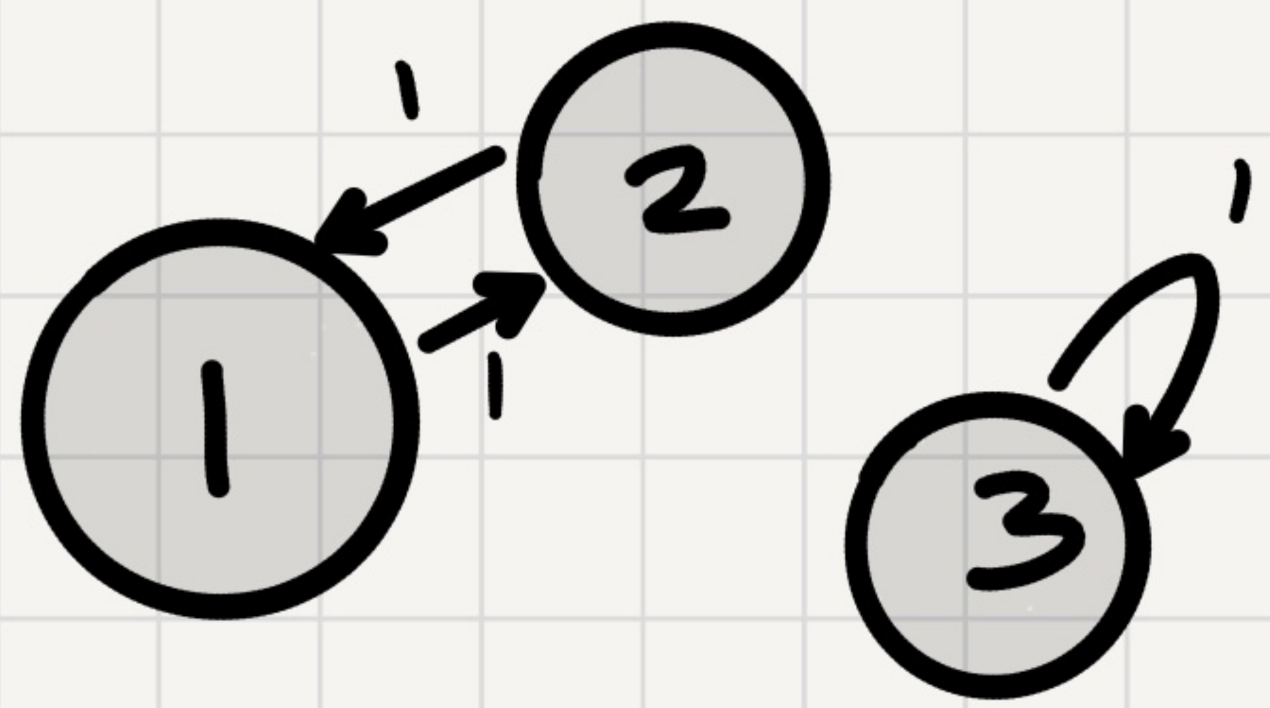


(D)



Theorem:

Any finite irreducible MC has one and only one stationary distribution.

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Theorem 2: (Long Term Fraction of Time in States)

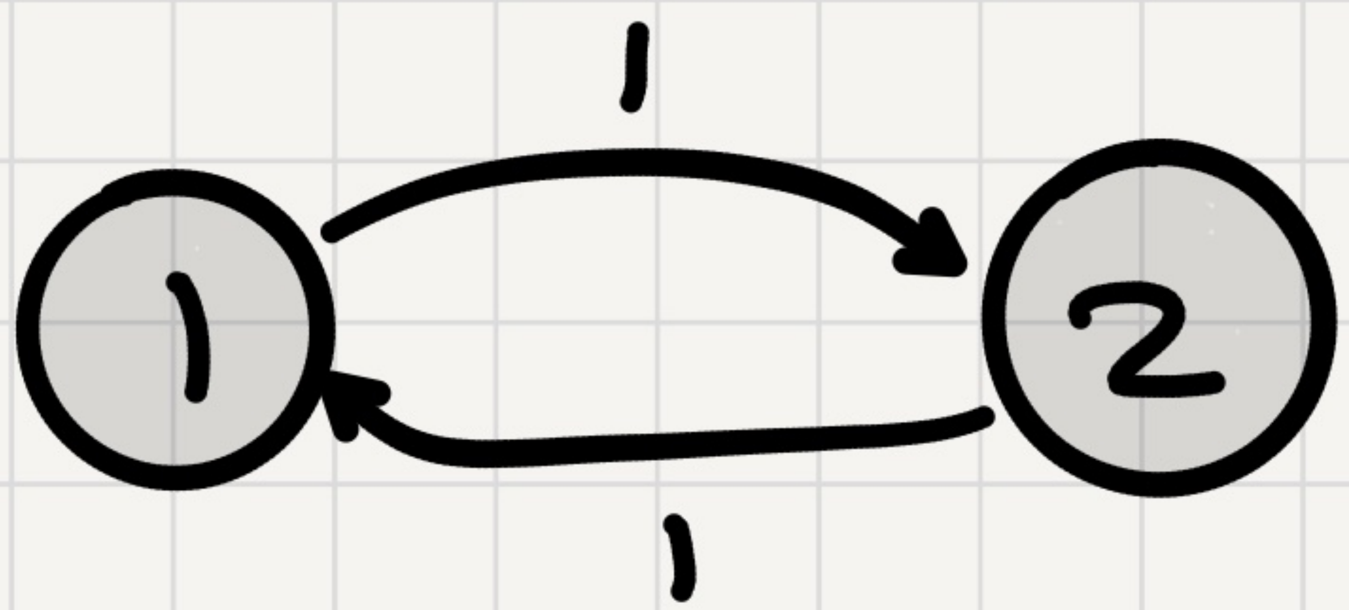
If $(X_n)_{n=0}^{\infty}$ is an irreducible MC on $\{1, \dots, k\}$ with stationary distribution π.

Then, for any start dist. π_0, for all $i \in \{1, \dots k\}$

$$\frac{1}{n} \sum_{m=0}^{n-1} \mathbb{1}\{X_m = i\} \xrightarrow[n \to \infty]{} \pi(i).$$

Converges to the Stationary Distribution

Example:



The MC is irreducible.

It's stationary dist satisfies

$$(\pi(1) \ \ \pi(2)) = (\pi(1) \ \ \pi(2)) \cdot \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\Rightarrow \pi(1) = \pi(2) = 1/2.$$

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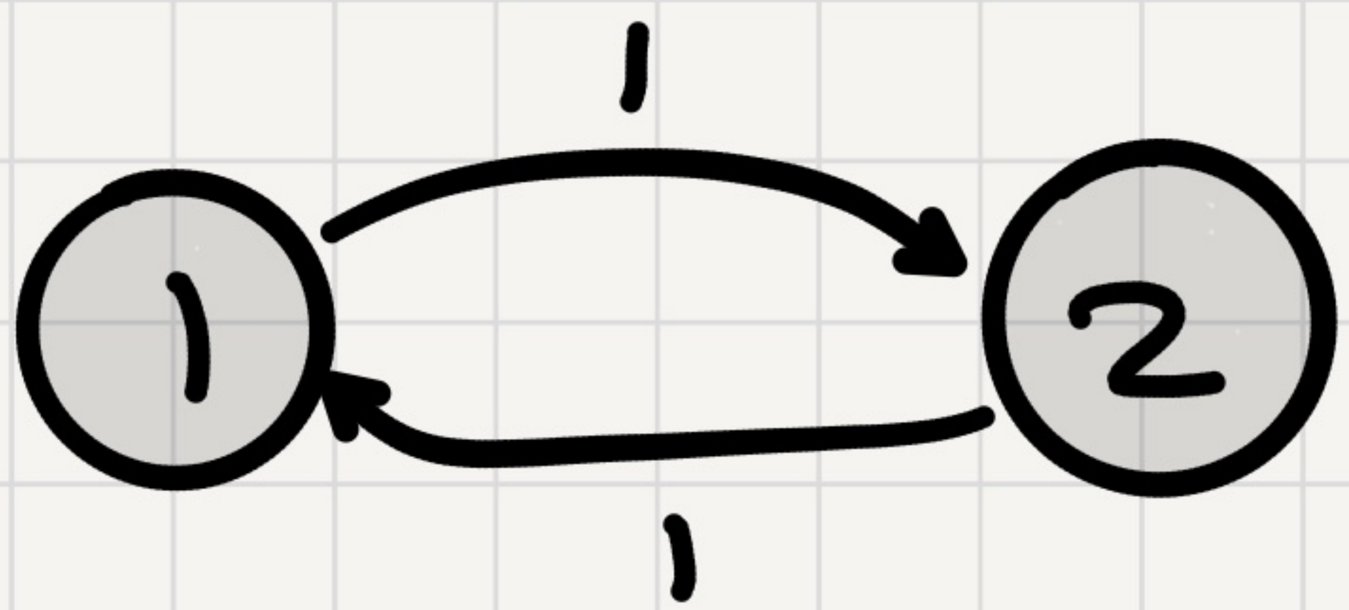

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But starting from 1: $\pi_0 = (1 \ 0)$

$\pi_1 = (0 \ 1)$

$\pi_2 = (1 \ 0) \ \ldots$

$\pi_{2m} = (1 \ 0)$

$\pi_{2m+1} = (0 \ 1)$

Periodicity

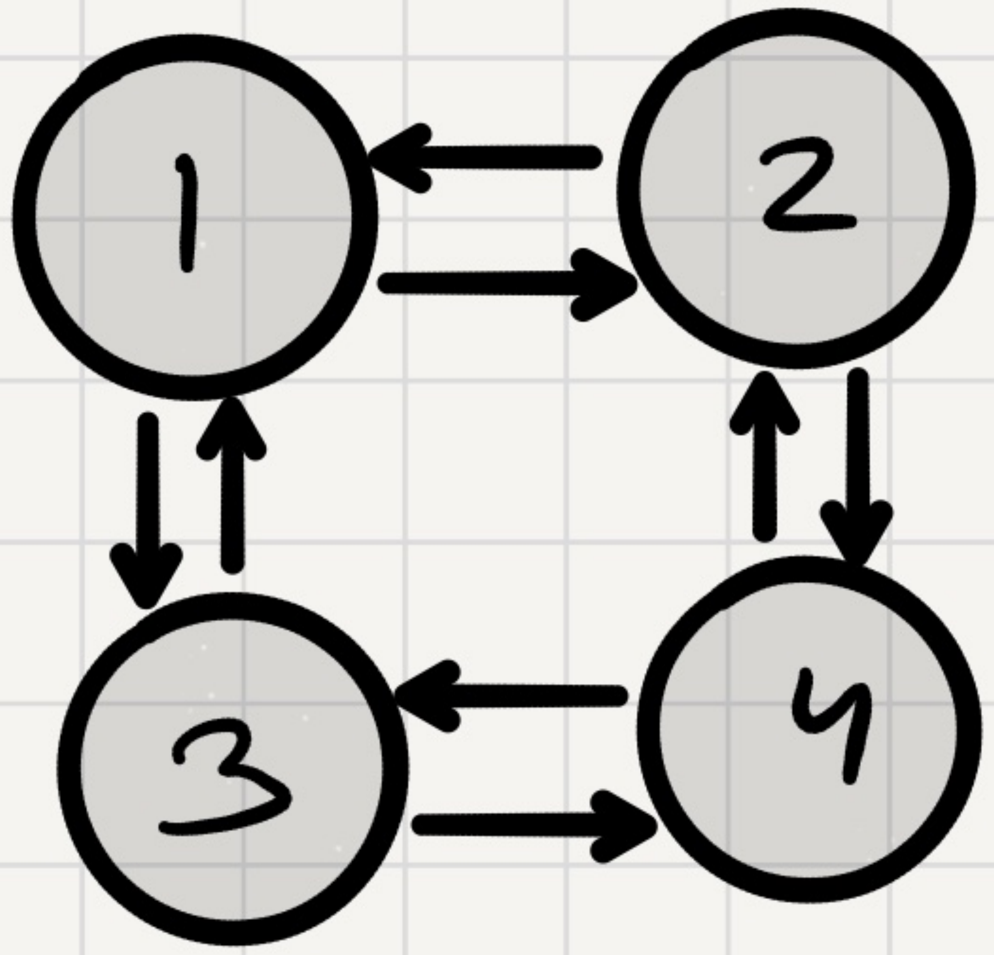
Def'n: The periodicity of a Markov Chain with transition matrix P is the gcd of lengths of all closed walks in the chain

$$\gcd(n > 0 \mid \exists i \text{ s.t. } P^n(i,i) > 0)$$

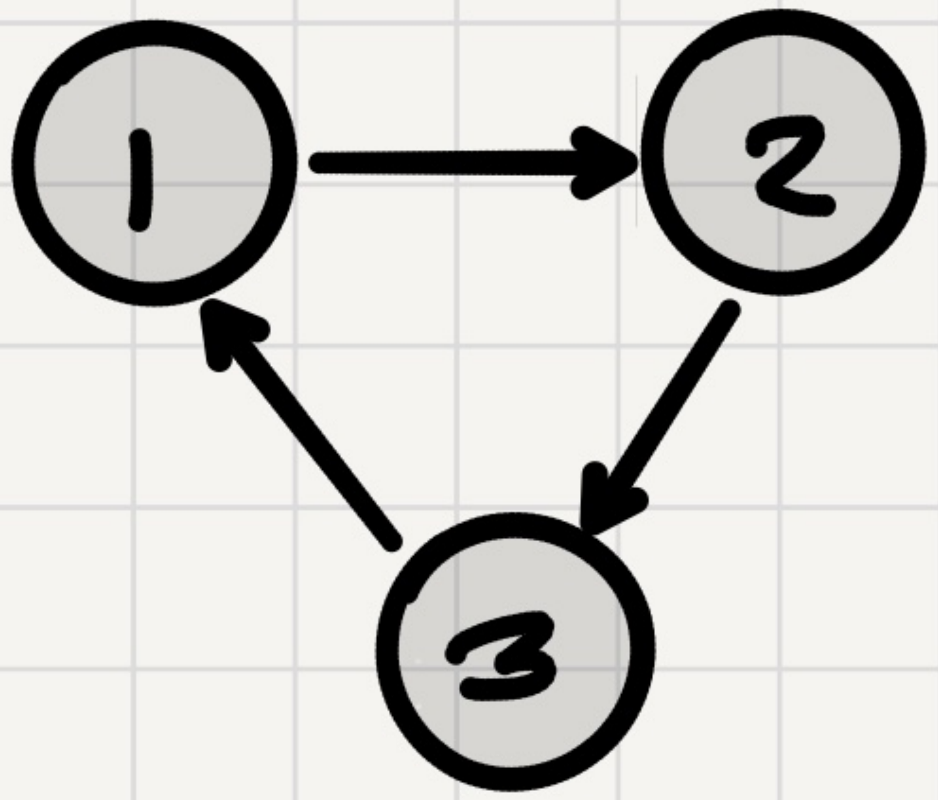
A Markov chain is **aperiodic** if this gcd $= 1$.

Which Markov Chains are Aperiodic?

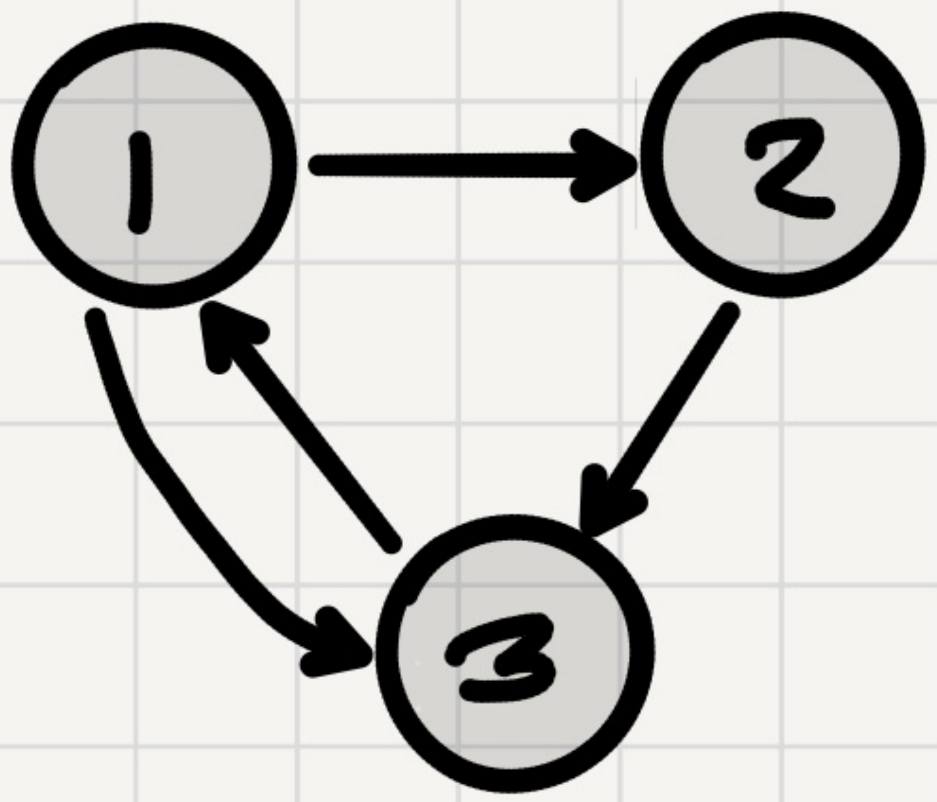
A.



B.



C.



Theorem: Let X_n be an irreducible aperiodic Markov Chain with stationary dist π.

Then, no matter what's the starting dist. π_0

$$\forall i: \quad \pi_n(i) \xrightarrow[n\to\infty]{} \pi(i).$$

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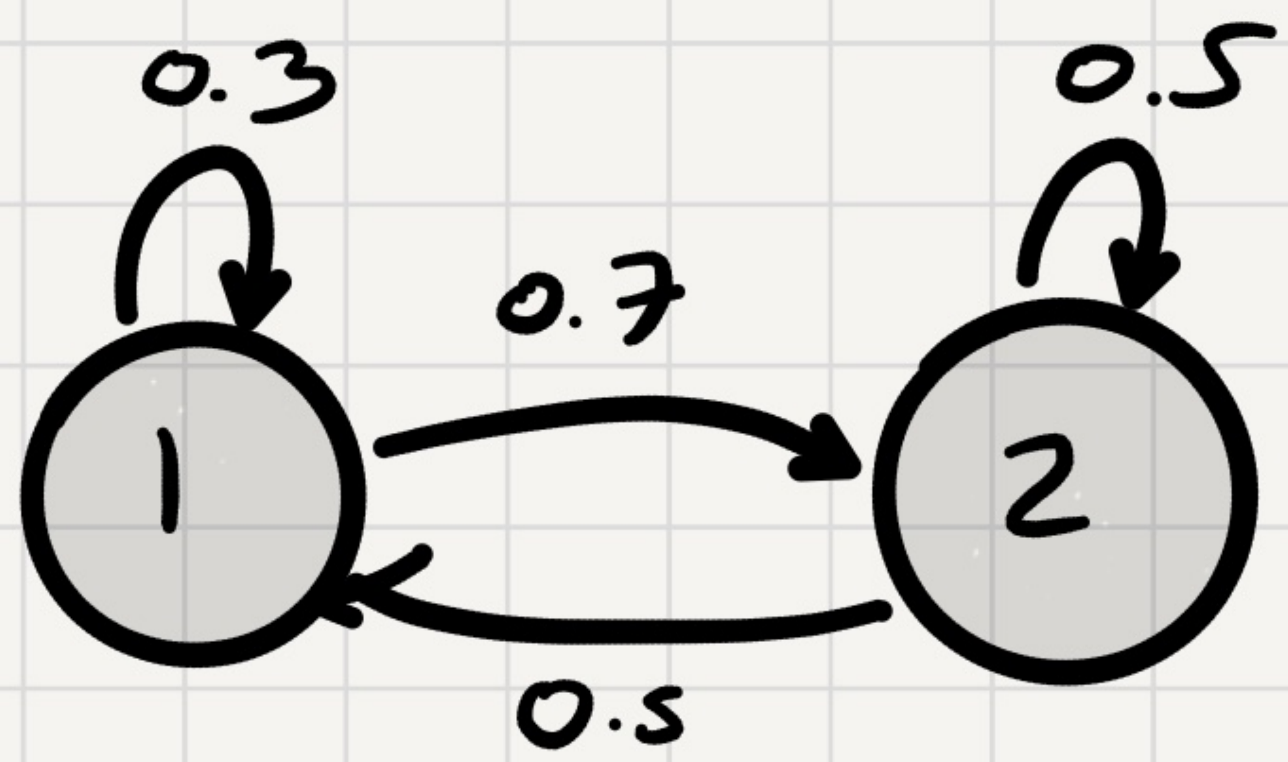

The stationary dist is $\pi = \left[\frac{0.5}{1.2}, \frac{0.7}{1.2}\right]$

$$\pi_n \xrightarrow[n\to\infty]{} \pi$$