To direct or not to direct.

Thm: n^2 is even $\implies$ n is even.

To direct or not to direct.

Thm: n^2 is even $\implies$ n is even.

Contrapositive Proof: $n = 2k + 1$, $n^2 = 4k^2 + 2k + 1$, $n^= 2(2k^2 + k) + 1$.

To direct or not to direct.

Thm: n^2 is even $\implies$ n is even.

Contrapositive Proof: $n = 2k+1$, $n^2 = 4k^2+2k+1$, $n^= 2(2k^2+k)+1$.

n is odd $\implies$ n^2 is odd.

To direct or not to direct.

Thm: n^2 is even $\implies$ n is even.

Contrapositive Proof: $n = 2k+1$, $n^2 = 4k^2 + 2k + 1$, $n^{=}2(2k^2 + k) + 1$.

n is odd $\implies$ n^2 is odd.

Direct Proof: $n^2 + n = n(n+1)$ which indicates it is even.

To direct or not to direct.

Thm: n^2 is even $\implies$ n is even.

Contrapositive Proof: $n = 2k+1$, $n^2 = 4k^2+2k+1$, $n^= 2(2k^2+k)+1$.

n is odd $\implies$ n^2 is odd.

Direct Proof: $n^2+n = n(n+1)$ which indicates it is even.

$n = n^2+n-n^2 = even + even - even$ so n is even.

To direct or not to direct.

Thm: n^2 is even $\implies$ n is even.

Contrapositive Proof: $n = 2k+1$, $n^2 = 4k^2+2k+1$, $n^= 2(2k^2+k)+1$.

n is odd $\implies$ n^2 is odd.

Direct Proof: $n^2+n = n(n+1)$ which indicates it is even.

$n = n^2+n-n^2 = even+even-even$ so n is even.

More detail: $even+even-even = 2q+2k-2m = 2(q+k-m)$.

How to: “trick”.

Theorem: $\forall n \in D_3, (11|n) \implies (11|\text{alt. sum of digits of } n)$

How to: “trick”.

Theorem: $\forall n \in D_3, (11|n) \implies (11|\text{alt. sum of digits of } n)$

Recall:

$n = 100a + 10b + c = 11k.$
alternate sum $= a - b + c$
want: $a - b + c = 11k'.$

How to: “trick”.

Theorem: $\forall n \in D_3, (11|n) \implies (11|\text{alt. sum of digits of } n)$

Recall:

$n = 100a + 10b + c = 11k$.

alternate sum $= a - b + c$

want: $a - b + c = 11k'$.

Proof:

How to: “trick”.

Theorem: $\forall n \in D_3, (11|n) \implies (11|\text{alt. sum of digits of } n)$

Recall:

$n = 100a + 10b + c = 11k$.
alternate sum $= a - b + c$
want: $a - b + c = 11k'$.

Proof: Assume $11|n$.

How to: “trick”.

Theorem: $\forall n \in D_3, (11|n) \implies (11|\text{alt. sum of digits of } n)$

Recall:

$n = 100a + 10b + c = 11k$.

alternate sum $= a - b + c$

want: $a - b + c = 11k'$.

Proof: Assume $11|n$.

$n = 100a + 10b + c = 11k$

How to: "trick".

Theorem: $\forall n \in D_3, (11|n) \implies (11|\text{alt. sum of digits of } n)$

Recall:

$n = 100a + 10b + c = 11k$.

alternate sum $= a - b + c$

want: $a - b + c = 11k'$.

Proof: Assume $11|n$.

$n = 100a + 10b + c = 11k \implies$

$99a + 11b + (a - b + c) = 11k$

How to: “trick”.

Theorem: $\forall n \in D_3, (11|n) \implies (11|\text{alt. sum of digits of } n)$

Recall:

$n = 100a + 10b + c = 11k.$

alternate sum $= a - b + c$

want: $a - b + c = 11k'.$

Proof: Assume $11|n$.

$$n = 100a + 10b + c = 11k \implies$$
$$99a + 11b + (a - b + c) = 11k \implies$$
$$a - b + c = 11k - 99a - 11b$$

How to: "trick".

Theorem: $\forall n \in D_3, (11|n) \implies (11|\text{alt. sum of digits of } n)$

Recall:

$n = 100a + 10b + c = 11k.$

alternate sum $= a - b + c$

want: $a - b + c = 11k'$.

Proof: Assume $11|n$.

$$n = 100a + 10b + c = 11k \implies$$
$$99a + 11b + (a - b + c) = 11k \implies$$
$$a - b + c = 11k - 99a - 11b \implies$$
$$a - b + c = 11(k - 9a - b)$$

How to: "trick".

Theorem: $\forall n \in D_3, (11|n) \implies (11|\text{alt. sum of digits of } n)$

Recall:

$n = 100a + 10b + c = 11k$.

alternate sum $= a - b + c$

want: $a - b + c = 11k'$.

Proof: Assume $11|n$.

$$n = 100a + 10b + c = 11k \implies$$
$$99a + 11b + (a - b + c) = 11k \implies$$
$$a - b + c = 11k - 99a - 11b \implies$$
$$a - b + c = 11(k - 9a - b) \implies$$
$$a - b + c = 11\ell$$

How to: "trick".

Theorem: $\forall n \in D_3, (11|n) \implies (11|\text{alt. sum of digits of } n)$

Recall:

$n = 100a + 10b + c = 11k$.
alternate sum $= a - b + c$
want: $a - b + c = 11k'$.

Proof: Assume $11|n$.

$$n = 100a + 10b + c = 11k \implies$$
$$99a + 11b + (a - b + c) = 11k \implies$$
$$a - b + c = 11k - 99a - 11b \implies$$
$$a - b + c = 11(k - 9a - b) \implies$$
$$a - b + c = 11\ell \text{ where } \ell = (k - 9a - b) \in Z$$

How to: "trick".

Theorem: $\forall n \in D_3, (11|n) \implies (11|\text{alt. sum of digits of } n)$

Recall:

$n = 100a + 10b + c = 11k.$

alternate sum $= a - b + c$

want: $a - b + c = 11k'$.

Proof: Assume $11|n$.

$$n = 100a + 10b + c = 11k \implies$$
$$99a + 11b + (a - b + c) = 11k \implies$$
$$a - b + c = 11k - 99a - 11b \implies$$
$$a - b + c = 11(k - 9a - b) \implies$$
$$a - b + c = 11\ell \text{ where } \ell = (k - 9a - b) \in Z$$

That is 11|alternating sum of digits. □

CS70: Note 3. Induction!

Poll. What's the biggest number?

(A) 100

(B) 101

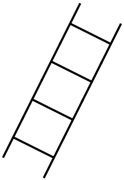
(C) n+1

(D) infinity.

(E) This is about the "recursive leap of faith."

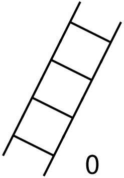
The natural numbers.

The natural numbers.



The natural numbers.

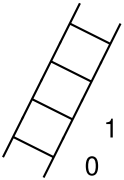
0,



0

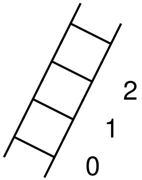
The natural numbers.

0, 1,



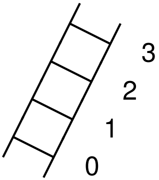
The natural numbers.

0, 1, 2,

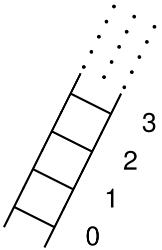


The natural numbers.

0, 1, 2, 3,



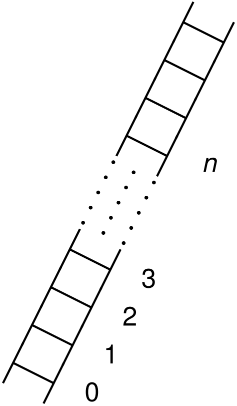
The natural numbers.



0, 1, 2, 3,

. . . ,

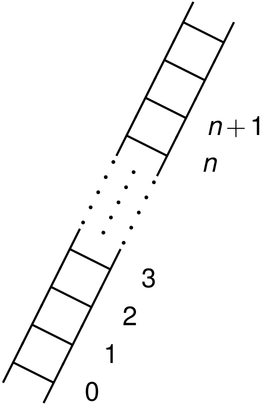
The natural numbers.



0, 1, 2, 3,

…, n,

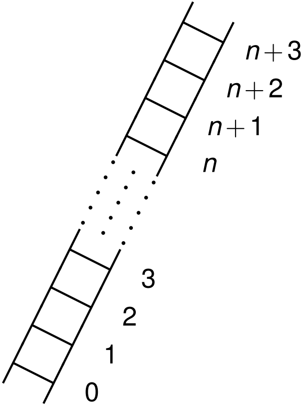
The natural numbers.



0, 1, 2, 3,

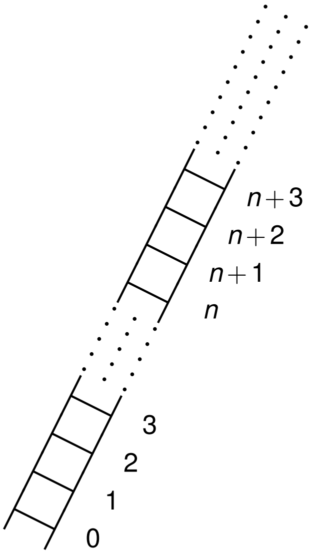
$\ldots$, n, $n+1$,

The natural numbers.



$0, 1, 2, 3,$
$\ldots, n, n+1, n+2, n+3,$

The natural numbers.



$0, 1, 2, 3,$

$\ldots, n, n+1, n+2, n+3, \ldots$

A formula.

A formula.

Teacher: Hello class.

A formula.

Teacher: Hello class.
Teacher:

A formula.

Teacher: Hello class.
Teacher: Please add the numbers from 1 to 100.

A formula.

Teacher: Hello class.
Teacher: Please add the numbers from 1 to 100.

Gauss: It's

A formula.

Teacher: Hello class.
Teacher: Please add the numbers from 1 to 100.

Gauss: It's $\frac{(100)(101)}{2}$

A formula.

Teacher: Hello class.
Teacher: Please add the numbers from 1 to 100.

Gauss: It's $\frac{(100)(101)}{2}$ or 5050!

A formula.

Teacher: Hello class.
Teacher: Please add the numbers from 1 to 100.

Gauss: It's $\frac{(100)(101)}{2}$ or 5050!

Five year old Gauss Theorem: $\forall(n \in N) : \sum_{i=0}^{n} i = \frac{(n)(n+1)}{2}$.

A formula.

Teacher: Hello class.
Teacher: Please add the numbers from 1 to 100.

Gauss: It's $\frac{(100)(101)}{2}$ or 5050!

Five year old Gauss Theorem: $\forall(n \in N) : \sum_{i=0}^{n} i = \frac{(n)(n+1)}{2}$.

It is a statement about all natural numbers.

A formula.

Teacher: Hello class.
Teacher: Please add the numbers from 1 to 100.

Gauss: It's $\frac{(100)(101)}{2}$ or 5050!

Five year old Gauss Theorem: $\forall(n \in N) : \sum_{i=0}^{n} i = \frac{(n)(n+1)}{2}$.

It is a statement about all natural numbers.

$\forall(n \in N) : P(n)$.

A formula.

Teacher: Hello class.
Teacher: Please add the numbers from 1 to 100.

Gauss: It's $\frac{(100)(101)}{2}$ or 5050!

Five year old Gauss Theorem: $\forall(n \in N) : \sum_{i=0}^{n} i = \frac{(n)(n+1)}{2}$.

It is a statement about all natural numbers.

$\forall(n \in N) : P(n)$.

$P(n)$ is "$\sum_{i=0}^{n} i = \frac{(n)(n+1)}{2}$".

A formula.

Teacher: Hello class.
Teacher: Please add the numbers from 1 to 100.

Gauss: It's $\frac{(100)(101)}{2}$ or 5050!

Five year old Gauss Theorem: $\forall(n \in N) : \sum_{i=0}^{n} i = \frac{(n)(n+1)}{2}$.

It is a statement about all natural numbers.

$$\forall(n \in N) : P(n).$$

$P(n)$ is "$\sum_{i=0}^{n} i = \frac{(n)(n+1)}{2}$".

Principle of Induction:

- Prove $P(0)$.

A formula.

Teacher: Hello class.
Teacher: Please add the numbers from 1 to 100.

Gauss: It's $\frac{(100)(101)}{2}$ or 5050!

Five year old Gauss Theorem: $\forall(n \in N) : \sum_{i=0}^{n} i = \frac{(n)(n+1)}{2}$.

It is a statement about all natural numbers.

$$\forall(n \in N) : P(n).$$

$P(n)$ is "$\sum_{i=0}^{n} i = \frac{(n)(n+1)}{2}$".

Principle of Induction:

- Prove $P(0)$.
- For $k \in \mathbb{N}$ Assume $P(k)$, "Induction Hypothesis"

A formula.

Teacher: Hello class.
Teacher: Please add the numbers from 1 to 100.

Gauss: It's $\frac{(100)(101)}{2}$ or 5050!

Five year old Gauss Theorem: $\forall(n \in N) : \sum_{i=0}^{n} i = \frac{(n)(n+1)}{2}$.

It is a statement about all natural numbers.

$\forall(n \in N) : P(n)$.

$P(n)$ is "$\sum_{i=0}^{n} i = \frac{(n)(n+1)}{2}$".

Principle of Induction:

- Prove $P(0)$.
- For $k \in \mathbb{N}$ Assume $P(k)$, "Induction Hypothesis"
- Prove $P(k+1)$. "Induction Step."

A formula.

Teacher: Hello class.
Teacher: Please add the numbers from 1 to 100.

Gauss: It's $\frac{(100)(101)}{2}$ or 5050!

Five year old Gauss Theorem: $\forall(n \in N) : \sum_{i=0}^{n} i = \frac{(n)(n+1)}{2}$.

It is a statement about all natural numbers.

$\forall(n \in N) : P(n)$.

$P(n)$ is "$\sum_{i=0}^{n} i = \frac{(n)(n+1)}{2}$".

Principle of Induction:

- Prove $P(0)$.
- For $k \in \mathbb{N}$ Assume $P(k)$, "Induction Hypothesis"
- Prove $P(k+1)$. "Induction Step."
- $\implies P(n)$ is true for all $n \in \mathbb{N}$.

Gauss induction proof.

Theorem: For all natural numbers n, $0+1+2\cdots n = \frac{n(n+1)}{2}$

Gauss induction proof.

Theorem: For all natural numbers n, $0 + 1 + 2 \cdots n = \frac{n(n+1)}{2}$

Base Case: $P(0)$: Does 0 = $\frac{0(0+1)}{2}$?

Gauss induction proof.

Theorem: For all natural numbers n, $0 + 1 + 2 \cdots n = \frac{n(n+1)}{2}$

Base Case: $P(0)$: Does 0 = $\frac{0(0+1)}{2}$? Yes.

Gauss induction proof.

Theorem: For all natural numbers n, $0+1+2\cdots n = \frac{n(n+1)}{2}$

Base Case: $P(0)$: Does 0 = $\frac{0(0+1)}{2}$? Yes.

Induction Step: Show $\forall k \geq 0, P(k) \implies P(k+1)$

Gauss induction proof.

Theorem: For all natural numbers n, $0+1+2\cdots n=\frac{n(n+1)}{2}$

Base Case: $P(0)$: Does 0 = $\frac{0(0+1)}{2}$? Yes.

Induction Step: Show $\forall k \geq 0, P(k) \implies P(k+1)$
Induction Hypothesis: $P(k) = 1+\cdots+k=\frac{k(k+1)}{2}$

Gauss induction proof.

Theorem: For all natural numbers n, $0+1+2\cdots n = \frac{n(n+1)}{2}$

Base Case: $P(0)$: Does 0 = $\frac{0(0+1)}{2}$? Yes.

Induction Step: Show $\forall k \geq 0, P(k) \implies P(k+1)$
Induction Hypothesis: $P(k) = 1+\cdots+k = \frac{k(k+1)}{2}$

$$1+\cdots+k+(k+1) \quad =$$

Gauss induction proof.

Theorem: For all natural numbers n, $0+1+2\cdots n = \frac{n(n+1)}{2}$

Base Case: $P(0)$: Does 0 = $\frac{0(0+1)}{2}$? Yes.

Induction Step: Show $\forall k \geq 0, P(k) \implies P(k+1)$
Induction Hypothesis: $P(k) = 1+\cdots+k = \frac{k(k+1)}{2}$

$$1+\cdots+k+(k+1) \quad = \quad \frac{k(k+1)}{2}+(k+1)$$

Gauss induction proof.

Theorem: For all natural numbers n, $0+1+2\cdots n = \frac{n(n+1)}{2}$

Base Case: $P(0)$: Does 0 = $\frac{0(0+1)}{2}$? Yes.

Induction Step: Show $\forall k \geq 0, P(k) \implies P(k+1)$
Induction Hypothesis: $P(k) = 1+\cdots+k = \frac{k(k+1)}{2}$

$$\begin{aligned} 1+\cdots+k+(k+1) &= \frac{k(k+1)}{2}+(k+1) \\ &= \frac{k^2+k+2(k+1)}{2} \end{aligned}$$

Gauss induction proof.

Theorem: For all natural numbers n, $0+1+2\cdots n = \frac{n(n+1)}{2}$

Base Case: $P(0)$: Does 0 = $\frac{0(0+1)}{2}$? Yes.

Induction Step: Show $\forall k \geq 0, P(k) \implies P(k+1)$
Induction Hypothesis: $P(k) = 1+\cdots+k = \frac{k(k+1)}{2}$

$$
\begin{aligned}
1+\cdots+k+(k+1) &= \frac{k(k+1)}{2}+(k+1) \\
&= \frac{k^2+k+2(k+1)}{2} \\
&= \frac{k^2+3k+2}{2}
\end{aligned}
$$

Gauss induction proof.

Theorem: For all natural numbers n, $0+1+2\cdots n = \frac{n(n+1)}{2}$

Base Case: $P(0)$: Does 0 = $\frac{0(0+1)}{2}$? Yes.

Induction Step: Show $\forall k \geq 0, P(k) \implies P(k+1)$
Induction Hypothesis: $P(k) = 1+\cdots+k = \frac{k(k+1)}{2}$

$$\begin{aligned}
1+\cdots+k+(k+1) &= \frac{k(k+1)}{2}+(k+1) \\
&= \frac{k^2+k+2(k+1)}{2} \\
&= \frac{k^2+3k+2}{2} \\
&= \frac{(k+1)(k+2)}{2}
\end{aligned}$$

Gauss induction proof.

Theorem: For all natural numbers n, $0+1+2\cdots n = \frac{n(n+1)}{2}$

Base Case: $P(0)$: Does $0 = \frac{0(0+1)}{2}$? Yes.

Induction Step: Show $\forall k \geq 0, P(k) \implies P(k+1)$

Induction Hypothesis: $P(k) = 1+\cdots+k = \frac{k(k+1)}{2}$

$$
\begin{aligned}
1+\cdots+k+(k+1) &= \frac{k(k+1)}{2}+(k+1) \\
&= \frac{k^2+k+2(k+1)}{2} \\
&= \frac{k^2+3k+2}{2} \\
&= \frac{(k+1)(k+2)}{2}
\end{aligned}
$$

$P(k+1)$!

Gauss induction proof.

Theorem: For all natural numbers n, $0+1+2\cdots n = \frac{n(n+1)}{2}$

Base Case: $P(0)$: Does 0 = $\frac{0(0+1)}{2}$? Yes.

Induction Step: Show $\forall k \geq 0, P(k) \implies P(k+1)$
Induction Hypothesis: $P(k) = 1+\cdots+k = \frac{k(k+1)}{2}$

$$\begin{aligned}
1+\cdots+k+(k+1) &= \frac{k(k+1)}{2}+(k+1) \\
&= \frac{k^2+k+2(k+1)}{2} \\
&= \frac{k^2+3k+2}{2} \\
&= \frac{(k+1)(k+2)}{2}
\end{aligned}$$

$P(k+1)$! By principle of induction...

Gauss induction proof.

Theorem: For all natural numbers n, $0+1+2\cdots n = \frac{n(n+1)}{2}$

Base Case: $P(0)$: Does 0 = $\frac{0(0+1)}{2}$? Yes.

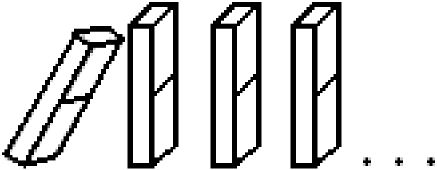
Induction Step: Show $\forall k \geq 0, P(k) \implies P(k+1)$
Induction Hypothesis: $P(k) = 1+\cdots+k = \frac{k(k+1)}{2}$

$$\begin{aligned}
1+\cdots+k+(k+1) &= \frac{k(k+1)}{2}+(k+1) \\
&= \frac{k^2+k+2(k+1)}{2} \\
&= \frac{k^2+3k+2}{2} \\
&= \frac{(k+1)(k+2)}{2}
\end{aligned}$$

$P(k+1)$! By principle of induction... □

Notes visualization

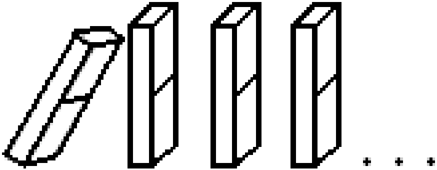
Note's visualization: an infinite sequence of dominos.



Prove they all fall down;

Notes visualization

Note's visualization: an infinite sequence of dominos.

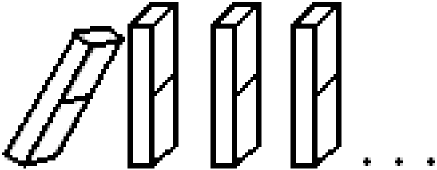


Prove they all fall down;

- $P(0)$ = "First domino falls"

Notes visualization

Note’s visualization: an infinite sequence of dominos.

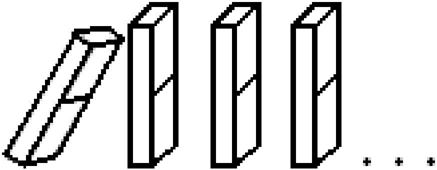


Prove they all fall down;

- $P(0)$ = “First domino falls”
- $(\forall k)\ P(k) \implies P(k+1)$:

Notes visualization

Note's visualization: an infinite sequence of dominos.

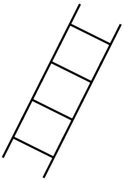


Prove they all fall down;

- $P(0)$ = "First domino falls"
- $(\forall k)\ P(k) \implies P(k+1)$:
  "kth domino falls implies that $k+1$st domino falls"

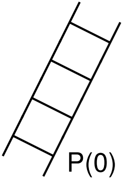
Climb an infinite ladder?

Climb an infinite ladder?



Climb an infinite ladder?

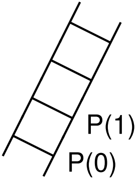
$P(0)$



Climb an infinite ladder?

$$P(0)$$

$$\forall k, P(k) \implies P(k+1)$$

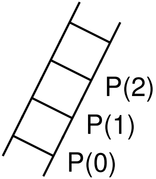


Climb an infinite ladder?

$$P(0)$$

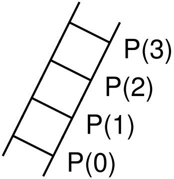
$$\forall k, P(k) \implies P(k+1)$$

$$P(0) \implies P(1) \implies P(2)$$

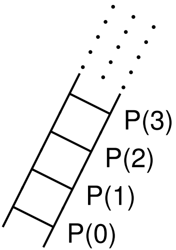


Climb an infinite ladder?

$$P(0)$$
$$\forall k, P(k) \implies P(k+1)$$
$$P(0) \implies P(1) \implies P(2) \implies P(3)$$

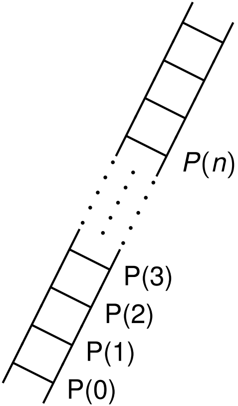


Climb an infinite ladder?



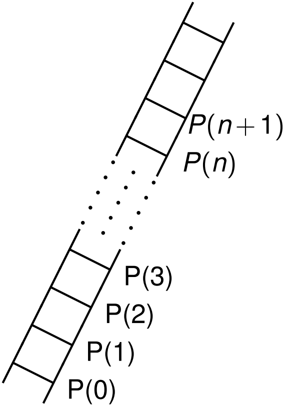
$$P(0)$$
$$\forall k, P(k) \implies P(k+1)$$
$$P(0) \implies P(1) \implies P(2) \implies P(3) \ldots$$

Climb an infinite ladder?



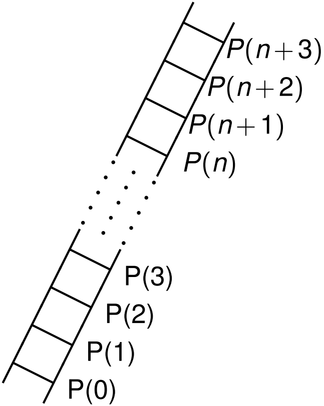
$$P(0)$$
$$\forall k, P(k) \implies P(k+1)$$
$$P(0) \implies P(1) \implies P(2) \implies P(3) \ldots$$

Climb an infinite ladder?



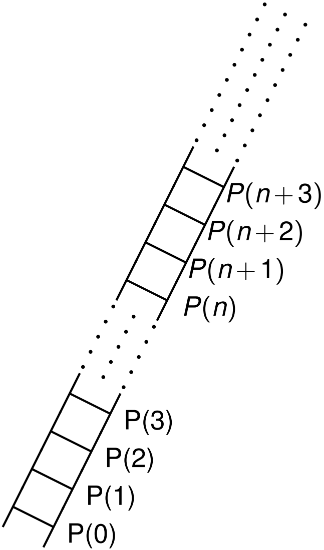
$$P(0)$$
$$\forall k, P(k) \implies P(k+1)$$
$$P(0) \implies P(1) \implies P(2) \implies P(3) \ldots$$

Climb an infinite ladder?



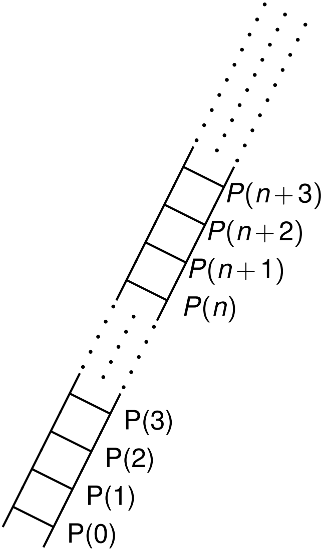
$$P(0)$$
$$\forall k, P(k) \implies P(k+1)$$
$$P(0) \implies P(1) \implies P(2) \implies P(3) \ldots$$

Climb an infinite ladder?



$$P(0)$$
$$\forall k, P(k) \implies P(k+1)$$
$$P(0) \implies P(1) \implies P(2) \implies P(3) \ldots$$
$$(\forall n \in N) P(n)$$

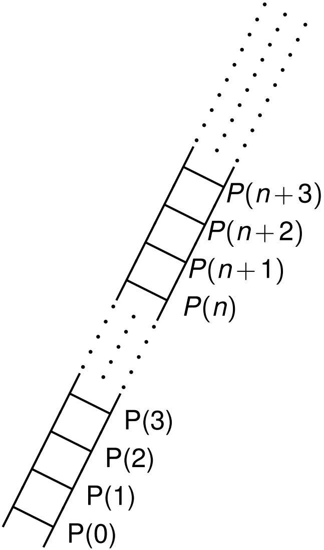
Climb an infinite ladder?



Your favorite example of forever..

$$P(0)$$
$$\forall k, P(k) \implies P(k+1)$$
$$P(0) \implies P(1) \implies P(2) \implies P(3) \ldots$$
$$(\forall n \in N) P(n)$$

Climb an infinite ladder?



$$P(0)$$
$$\forall k, P(k) \implies P(k+1)$$
$$P(0) \implies P(1) \implies P(2) \implies P(3) \dots$$
$$(\forall n \in N) P(n)$$

Your favorite example of forever..or the natural numbers...

Induction

The canonical way of proving statements of the form

$$(\forall k \in N)(P(k))$$

Induction

The canonical way of proving statements of the form

$$(\forall k \in N)(P(k))$$

- For all natural numbers n, $1+2\cdots n = \frac{n(n+1)}{2}$.

Induction

The canonical way of proving statements of the form

$$(\forall k \in N)(P(k))$$

- For all natural numbers n, $1 + 2 \cdots n = \frac{n(n+1)}{2}$.
- For all $n \in N$, $n^3 - n$ is divisible by 3.

Induction

The canonical way of proving statements of the form

$$(\forall k \in N)(P(k))$$

- For all natural numbers n, $1 + 2 \cdots n = \frac{n(n+1)}{2}$.
- For all $n \in N$, $n^3 - n$ is divisible by 3.
- The sum of the first n odd integers is a perfect square.

Induction

The canonical way of proving statements of the form

$$(\forall k \in N)(P(k))$$

- For all natural numbers n, $1+2\cdots n = \frac{n(n+1)}{2}$.
- For all $n \in N$, $n^3 - n$ is divisible by 3.
- The sum of the first n odd integers is a perfect square.

The basic form

Induction

The canonical way of proving statements of the form

$$(\forall k \in N)(P(k))$$

- For all natural numbers n, $1+2\cdots n = \frac{n(n+1)}{2}$.
- For all $n \in N$, $n^3 - n$ is divisible by 3.
- The sum of the first n odd integers is a perfect square.

The basic form

- Prove $P(0)$. “Base Case”.

Induction

The canonical way of proving statements of the form

$$(\forall k \in N)(P(k))$$

- For all natural numbers n, $1+2\cdots n = \frac{n(n+1)}{2}$.
- For all $n \in N$, $n^3 - n$ is divisible by 3.
- The sum of the first n odd integers is a perfect square.

The basic form

- Prove $P(0)$. “Base Case”.
- $P(k) \implies P(k+1)$

Induction

The canonical way of proving statements of the form

$$(\forall k \in N)(P(k))$$

- For all natural numbers n, $1 + 2 \cdots n = \frac{n(n+1)}{2}$.
- For all $n \in N$, $n^3 - n$ is divisible by 3.
- The sum of the first n odd integers is a perfect square.

The basic form

- Prove $P(0)$. “Base Case”.
- $P(k) \implies P(k+1)$
  - Assume $P(k)$, “Induction Hypothesis”

Induction

The canonical way of proving statements of the form

$$(\forall k \in N)(P(k))$$

- For all natural numbers n, $1 + 2 \cdots n = \frac{n(n+1)}{2}$.
- For all $n \in N$, $n^3 - n$ is divisible by 3.
- The sum of the first n odd integers is a perfect square.

The basic form

- Prove $P(0)$. “Base Case”.
- $P(k) \implies P(k+1)$
  - Assume $P(k)$, “Induction Hypothesis”
  - Prove $P(k+1)$. “Induction Step.”

Induction

The canonical way of proving statements of the form

$$(\forall k \in N)(P(k))$$

- For all natural numbers n, $1 + 2 \cdots n = \frac{n(n+1)}{2}$.
- For all $n \in N$, $n^3 - n$ is divisible by 3.
- The sum of the first n odd integers is a perfect square.

The basic form

- Prove $P(0)$. “Base Case”.
- $P(k) \implies P(k+1)$
  - Assume $P(k)$, “Induction Hypothesis”
  - Prove $P(k+1)$. “Induction Step.”

$P(n)$ true for all natural numbers n!!!

Induction

The canonical way of proving statements of the form

$$(\forall k \in N)(P(k))$$

- For all natural numbers n, $1 + 2 \cdots n = \frac{n(n+1)}{2}$.
- For all $n \in N$, $n^3 - n$ is divisible by 3.
- The sum of the first n odd integers is a perfect square.

The basic form

- Prove $P(0)$. "Base Case".
- $P(k) \implies P(k+1)$
  - Assume $P(k)$, "Induction Hypothesis"
  - Prove $P(k+1)$. "Induction Step."

$P(n)$ true for all natural numbers n!!!
Get to use $P(k)$ to prove $P(k+1)$!

Induction

The canonical way of proving statements of the form

$$(\forall k \in N)(P(k))$$

- For all natural numbers n, $1+2\cdots n = \frac{n(n+1)}{2}$.
- For all $n \in N$, $n^3 - n$ is divisible by 3.
- The sum of the first n odd integers is a perfect square.

The basic form

- Prove $P(0)$. “Base Case”.
- $P(k) \implies P(k+1)$
  - Assume $P(k)$, “Induction Hypothesis”
  - Prove $P(k+1)$. “Induction Step.”

$P(n)$ true for all natural numbers n!!!
Get to use $P(k)$ to prove $P(k+1)$! !

Induction

The canonical way of proving statements of the form

$$(\forall k \in N)(P(k))$$

- For all natural numbers n, $1+2\cdots n = \frac{n(n+1)}{2}$.
- For all $n \in N$, $n^3 - n$ is divisible by 3.
- The sum of the first n odd integers is a perfect square.

The basic form

- Prove $P(0)$. "Base Case".
- $P(k) \implies P(k+1)$
  - Assume $P(k)$, "Induction Hypothesis"
  - Prove $P(k+1)$. "Induction Step."

$P(n)$ true for all natural numbers n!!!
Get to use $P(k)$ to prove $P(k+1)$! ! !

Induction

The canonical way of proving statements of the form

$$(\forall k \in N)(P(k))$$

- For all natural numbers n, $1+2\cdots n = \frac{n(n+1)}{2}$.
- For all $n \in N$, $n^3 - n$ is divisible by 3.
- The sum of the first n odd integers is a perfect square.

The basic form

- Prove $P(0)$. “Base Case”.
- $P(k) \implies P(k+1)$
  - Assume $P(k)$, “Induction Hypothesis”
  - Prove $P(k+1)$. “Induction Step.”

$P(n)$ true for all natural numbers n!!!
Get to use $P(k)$ to prove $P(k+1)$! ! ! !

Gauss and Induction

Child Gauss: $(\forall \mathbf{n} \in \mathbf{N})(\sum_{i=0}^{n} i = \frac{n(n+1)}{2})$

Gauss and Induction

Child Gauss: $(\forall \mathbf{n} \in \mathbf{N})(\sum_{i=0}^{n} i = \frac{n(n+1)}{2})$ Proof?

Gauss and Induction

Child Gauss: $(\forall \mathbf{n} \in \mathbf{N})(\sum_{i=0}^{n} i = \frac{n(n+1)}{2})$ Proof?

Idea: assume predicate $P(n)$ for $n = k$.

Gauss and Induction

Child Gauss: $(\forall \mathbf{n} \in \mathbf{N})(\sum_{i=0}^{n} i = \frac{n(n+1)}{2})$ Proof?

Idea: assume predicate $P(n)$ for $n = k$. $P(k)$ is $\sum_{i=0}^{k} i = \frac{k(k+1)}{2}$.

Gauss and Induction

Child Gauss: $(\forall \mathbf{n} \in \mathbf{N})(\sum_{i=0}^{n} i = \frac{n(n+1)}{2})$ Proof?

Idea: assume predicate $P(n)$ for $n = k$. $P(k)$ is $\sum_{i=0}^{k} i = \frac{k(k+1)}{2}$.

Is predicate, $P(n)$ true for $n = k + 1$?

Gauss and Induction

Child Gauss: $(\forall \mathbf{n} \in \mathbf{N})(\sum_{i=0}^{n} i = \frac{n(n+1)}{2})$ Proof?

Idea: assume predicate $P(n)$ for $n = k$. $P(k)$ is $\sum_{i=0}^{k} i = \frac{k(k+1)}{2}$.

Is predicate, $P(n)$ true for $n = k+1$?

$\sum_{i=0}^{k+1} i$

Gauss and Induction

Child Gauss: $(\forall \mathbf{n} \in \mathbf{N})(\sum_{i=0}^{n} i = \frac{n(n+1)}{2})$ Proof?

Idea: assume predicate $P(n)$ for $n = k$. $P(k)$ is $\sum_{i=0}^{k} i = \frac{k(k+1)}{2}$.

Is predicate, $P(n)$ true for $n = k+1$?

$\sum_{i=0}^{k+1} i = (\sum_{i=1}^{k} i) + (k+1)$

Gauss and Induction

Child Gauss: $(\forall \mathbf{n} \in \mathbf{N})(\sum_{i=0}^{n} i = \frac{n(n+1)}{2})$ Proof?

Idea: assume predicate $P(n)$ for $n = k$. $P(k)$ is $\sum_{i=0}^{k} i = \frac{k(k+1)}{2}$.

Is predicate, $P(n)$ true for $n = k+1$?

$\sum_{i=0}^{k+1} i = (\sum_{i=1}^{k} i) + (k+1) = \frac{k(k+1)}{2} + k + 1$

Gauss and Induction

Child Gauss: $(\forall \mathbf{n} \in \mathbf{N})(\sum_{i=0}^{n} i = \frac{n(n+1)}{2})$ Proof?

Idea: assume predicate $P(n)$ for $n = k$. $P(k)$ is $\sum_{i=0}^{k} i = \frac{k(k+1)}{2}$.

Is predicate, $P(n)$ true for $n = k+1$?

$\sum_{i=0}^{k+1} i = (\sum_{i=1}^{k} i) + (k+1) = \frac{k(k+1)}{2} + k + 1 = \frac{k(k+1)+2(k+1)}{2}$

Gauss and Induction

Child Gauss: $(\forall \mathbf{n} \in \mathbf{N})(\sum_{i=0}^{n} i = \frac{n(n+1)}{2})$ Proof?

Idea: assume predicate $P(n)$ for $n = k$. $P(k)$ is $\sum_{i=0}^{k} i = \frac{k(k+1)}{2}$.

Is predicate, $P(n)$ true for $n = k+1$?

$\sum_{i=0}^{k+1} i = (\sum_{i=1}^{k} i) + (k+1) = \frac{k(k+1)}{2} + k + 1 = \frac{k(k+1)+2(k+1)}{2} = \frac{(k+1)(k+2)}{2}$.

Gauss and Induction

Child Gauss: $(\forall \mathbf{n} \in \mathbf{N})(\sum_{i=0}^{n} i = \frac{n(n+1)}{2})$ Proof?

Idea: assume predicate $P(n)$ for $n = k$. $P(k)$ is $\sum_{i=0}^{k} i = \frac{k(k+1)}{2}$.

Is predicate, $P(n)$ true for $n = k+1$?

$\sum_{i=0}^{k+1} i = (\sum_{i=1}^{k} i) + (k+1) = \frac{k(k+1)}{2} + k + 1 = \frac{k(k+1)+2(k+1)}{2} = \frac{(k+1)(k+2)}{2}$.

How about $k+2$.

Gauss and Induction

Child Gauss: $(\forall \mathbf{n} \in \mathbf{N})(\sum_{i=0}^{n} i = \frac{n(n+1)}{2})$ Proof?

Idea: assume predicate $P(n)$ for $n = k$. $P(k)$ is $\sum_{i=0}^{k} i = \frac{k(k+1)}{2}$.

Is predicate, $P(n)$ true for $n = k+1$?

$\sum_{i=0}^{k+1} i = (\sum_{i=1}^{k} i) + (k+1) = \frac{k(k+1)}{2} + k + 1 = \frac{k(k+1)+2(k+1)}{2} = \frac{(k+1)(k+2)}{2}$.

How about $k+2$. Same argument starting at $k+1$ works!

Gauss and Induction

Child Gauss: $(\forall \mathbf{n} \in \mathbf{N})(\sum_{i=0}^{n} i = \frac{n(n+1)}{2})$ Proof?

Idea: assume predicate $P(n)$ for $n = k$. $P(k)$ is $\sum_{i=0}^{k} i = \frac{k(k+1)}{2}$.

Is predicate, $P(n)$ true for $n = k+1$?

$$\sum_{i=0}^{k+1} i = (\sum_{i=1}^{k} i) + (k+1) = \frac{k(k+1)}{2} + k + 1 = \frac{k(k+1)+2(k+1)}{2} = \frac{(k+1)(k+2)}{2}.$$

How about $k+2$. Same argument starting at $k+1$ works!

Induction Step.

Gauss and Induction

Child Gauss: $(\forall \mathbf{n} \in \mathbf{N})(\sum_{i=0}^{n} i = \frac{n(n+1)}{2})$ Proof?

Idea: assume predicate $P(n)$ for $n = k$. $P(k)$ is $\sum_{i=0}^{k} i = \frac{k(k+1)}{2}$.

Is predicate, $P(n)$ true for $n = k+1$?

$$\sum_{i=0}^{k+1} i = (\sum_{i=1}^{k} i) + (k+1) = \frac{k(k+1)}{2} + k + 1 = \frac{k(k+1)+2(k+1)}{2} = \frac{(k+1)(k+2)}{2}.$$

How about $k+2$. Same argument starting at $k+1$ works!

Induction Step. $P(k) \implies P(k+1)$.

Gauss and Induction

Child Gauss: $(\forall \mathbf{n} \in \mathbf{N})(\sum_{i=0}^{n} i = \frac{n(n+1)}{2})$ Proof?

Idea: assume predicate $P(n)$ for $n = k$. $P(k)$ is $\sum_{i=0}^{k} i = \frac{k(k+1)}{2}$.

Is predicate, $P(n)$ true for $n = k+1$?

$\sum_{i=0}^{k+1} i = (\sum_{i=1}^{k} i) + (k+1) = \frac{k(k+1)}{2} + k + 1 = \frac{k(k+1)+2(k+1)}{2} = \frac{(k+1)(k+2)}{2}$.

How about $k+2$. Same argument starting at $k+1$ works!

Induction Step. $P(k) \implies P(k+1)$.

Is this a proof?

Gauss and Induction

Child Gauss: $(\forall \mathbf{n} \in \mathbf{N})(\sum_{i=0}^{n} i = \frac{n(n+1)}{2})$ Proof?

Idea: assume predicate $P(n)$ for $n = k$. $P(k)$ is $\sum_{i=0}^{k} i = \frac{k(k+1)}{2}$.

Is predicate, $P(n)$ true for $n = k+1$?

$$\sum_{i=0}^{k+1} i = (\sum_{i=1}^{k} i) + (k+1) = \frac{k(k+1)}{2} + k + 1 = \frac{k(k+1)+2(k+1)}{2} = \frac{(k+1)(k+2)}{2}.$$

How about $k+2$. Same argument starting at $k+1$ works!

Induction Step. $P(k) \implies P(k+1)$.

Is this a proof? It shows that we can always move to the next step.

Gauss and Induction

Child Gauss: $(\forall \mathbf{n} \in \mathbf{N})(\sum_{i=0}^{n} i = \frac{n(n+1)}{2})$ Proof?

Idea: assume predicate $P(n)$ for $n = k$. $P(k)$ is $\sum_{i=0}^{k} i = \frac{k(k+1)}{2}$.

Is predicate, $P(n)$ true for $n = k+1$?

$$\sum_{i=0}^{k+1} i = (\sum_{i=1}^{k} i) + (k+1) = \frac{k(k+1)}{2} + k + 1 = \frac{k(k+1)+2(k+1)}{2} = \frac{(k+1)(k+2)}{2}.$$

How about $k+2$. Same argument starting at $k+1$ works!

Induction Step. $P(k) \implies P(k+1)$.

Is this a proof? It shows that we can always move to the next step.

Need to start somewhere.

Gauss and Induction

Child Gauss: $(\forall \mathbf{n} \in \mathbf{N})(\sum_{i=0}^{n} i = \frac{n(n+1)}{2})$ Proof?

Idea: assume predicate $P(n)$ for $n = k$. $P(k)$ is $\sum_{i=0}^{k} i = \frac{k(k+1)}{2}$.

Is predicate, $P(n)$ true for $n = k+1$?

$\sum_{i=0}^{k+1} i = (\sum_{i=1}^{k} i) + (k+1) = \frac{k(k+1)}{2} + k + 1 = \frac{k(k+1)+2(k+1)}{2} = \frac{(k+1)(k+2)}{2}$.

How about $k+2$. Same argument starting at $k+1$ works!

Induction Step. $P(k) \implies P(k+1)$.

Is this a proof? It shows that we can always move to the next step.

Need to start somewhere. $P(0)$ is $\sum_{i=0}^{0} i = \frac{(0)(0+1)}{2}$

Gauss and Induction

Child Gauss: $(\forall \mathbf{n} \in \mathbf{N})(\sum_{i=0}^{n} i = \frac{n(n+1)}{2})$ Proof?

Idea: assume predicate $P(n)$ for $n = k$. $P(k)$ is $\sum_{i=0}^{k} i = \frac{k(k+1)}{2}$.

Is predicate, $P(n)$ true for $n = k+1$?

$$\sum_{i=0}^{k+1} i = (\sum_{i=1}^{k} i) + (k+1) = \frac{k(k+1)}{2} + k + 1 = \frac{k(k+1)+2(k+1)}{2} = \frac{(k+1)(k+2)}{2}.$$

How about $k+2$. Same argument starting at $k+1$ works!

Induction Step. $P(k) \implies P(k+1)$.

Is this a proof? It shows that we can always move to the next step.

Need to start somewhere. $P(0)$ is $\sum_{i=0}^{0} i = \frac{(0)(0+1)}{2}$ **Base Case.**

Gauss and Induction

Child Gauss: $(\forall \mathbf{n} \in \mathbf{N})(\sum_{i=0}^{n} i = \frac{n(n+1)}{2})$ Proof?

Idea: assume predicate $P(n)$ for $n = k$. $P(k)$ is $\sum_{i=0}^{k} i = \frac{k(k+1)}{2}$.

Is predicate, $P(n)$ true for $n = k+1$?

$$\sum_{i=0}^{k+1} i = (\sum_{i=1}^{k} i) + (k+1) = \frac{k(k+1)}{2} + k + 1 = \frac{k(k+1)+2(k+1)}{2} = \frac{(k+1)(k+2)}{2}.$$

How about $k+2$. Same argument starting at $k+1$ works!

Induction Step. $P(k) \implies P(k+1)$.

Is this a proof? It shows that we can always move to the next step.

Need to start somewhere. $P(0)$ is $\sum_{i=0}^{0} i = \frac{(0)(0+1)}{2}$ **Base Case.**

Statement is true for $n = 0$

Gauss and Induction

Child Gauss: $(\forall \mathbf{n} \in \mathbf{N})(\sum_{i=0}^{n} i = \frac{n(n+1)}{2})$ Proof?

Idea: assume predicate $P(n)$ for $n = k$. $P(k)$ is $\sum_{i=0}^{k} i = \frac{k(k+1)}{2}$.

Is predicate, $P(n)$ true for $n = k+1$?

$$\sum_{i=0}^{k+1} i = (\sum_{i=1}^{k} i) + (k+1) = \frac{k(k+1)}{2} + k + 1 = \frac{k(k+1)+2(k+1)}{2} = \frac{(k+1)(k+2)}{2}.$$

How about $k+2$. Same argument starting at $k+1$ works!

Induction Step. $P(k) \implies P(k+1)$.

Is this a proof? It shows that we can always move to the next step.

Need to start somewhere. $P(0)$ is $\sum_{i=0}^{0} i = \frac{(0)(0+1)}{2}$ **Base Case.**

Statement is true for $n = 0$ $P(0)$ is true

Gauss and Induction

Child Gauss: $(\forall \mathbf{n} \in \mathbf{N})(\sum_{i=0}^{n} i = \frac{n(n+1)}{2})$ Proof?

Idea: assume predicate $P(n)$ for $n = k$. $P(k)$ is $\sum_{i=0}^{k} i = \frac{k(k+1)}{2}$.

Is predicate, $P(n)$ true for $n = k+1$?

$$\sum_{i=0}^{k+1} i = (\sum_{i=1}^{k} i) + (k+1) = \frac{k(k+1)}{2} + k + 1 = \frac{k(k+1)+2(k+1)}{2} = \frac{(k+1)(k+2)}{2}.$$

How about $k+2$. Same argument starting at $k+1$ works!

Induction Step. $P(k) \implies P(k+1)$.

Is this a proof? It shows that we can always move to the next step.

Need to start somewhere. $P(0)$ is $\sum_{i=0}^{0} i = \frac{(0)(0+1)}{2}$ **Base Case.**

Statement is true for $n = 0$ $P(0)$ is true
plus inductive step

Gauss and Induction

Child Gauss: $(\forall \mathbf{n} \in \mathbf{N})(\sum_{i=0}^{n} i = \frac{n(n+1)}{2})$ Proof?

Idea: assume predicate $P(n)$ for $n = k$. $P(k)$ is $\sum_{i=0}^{k} i = \frac{k(k+1)}{2}$.

Is predicate, $P(n)$ true for $n = k+1$?

$$\sum_{i=0}^{k+1} i = (\sum_{i=1}^{k} i) + (k+1) = \frac{k(k+1)}{2} + k + 1 = \frac{k(k+1)+2(k+1)}{2} = \frac{(k+1)(k+2)}{2}.$$

How about $k+2$. Same argument starting at $k+1$ works!

Induction Step. $P(k) \implies P(k+1)$.

Is this a proof? It shows that we can always move to the next step.

Need to start somewhere. $P(0)$ is $\sum_{i=0}^{0} i = \frac{(0)(0+1)}{2}$ **Base Case.**

Statement is true for $n = 0$ $P(0)$ is true

plus inductive step $\implies$ true for $n = 1$

Gauss and Induction

Child Gauss: $(\forall \mathbf{n} \in \mathbf{N})(\sum_{i=0}^{n} i = \frac{n(n+1)}{2})$ Proof?

Idea: assume predicate $P(n)$ for $n = k$. $P(k)$ is $\sum_{i=0}^{k} i = \frac{k(k+1)}{2}$.

Is predicate, $P(n)$ true for $n = k+1$?

$$\sum_{i=0}^{k+1} i = (\sum_{i=1}^{k} i) + (k+1) = \frac{k(k+1)}{2} + k + 1 = \frac{k(k+1)+2(k+1)}{2} = \frac{(k+1)(k+2)}{2}.$$

How about $k+2$. Same argument starting at $k+1$ works!

Induction Step. $P(k) \implies P(k+1)$.

Is this a proof? It shows that we can always move to the next step.

Need to start somewhere. $P(0)$ is $\sum_{i=0}^{0} i = \frac{(0)(0+1)}{2}$ **Base Case.**

Statement is true for $n = 0$ $P(0)$ is true
plus inductive step $\implies$ true for $n = 1$ $(P(0) \wedge (P(0) \implies P(1))) \implies P(1)$

Gauss and Induction

Child Gauss: $(\forall \mathbf{n} \in \mathbf{N})(\sum_{i=0}^{n} i = \frac{n(n+1)}{2})$ Proof?

Idea: assume predicate $P(n)$ for $n = k$. $P(k)$ is $\sum_{i=0}^{k} i = \frac{k(k+1)}{2}$.

Is predicate, $P(n)$ true for $n = k+1$?

$$\sum_{i=0}^{k+1} i = (\sum_{i=1}^{k} i) + (k+1) = \frac{k(k+1)}{2} + k + 1 = \frac{k(k+1)+2(k+1)}{2} = \frac{(k+1)(k+2)}{2}.$$

How about $k+2$. Same argument starting at $k+1$ works!

Induction Step. $P(k) \implies P(k+1)$.

Is this a proof? It shows that we can always move to the next step.

Need to start somewhere. $P(0)$ is $\sum_{i=0}^{0} i = \frac{(0)(0+1)}{2}$ **Base Case.**

Statement is true for $n = 0$ $P(0)$ is true

plus inductive step $\implies$ true for $n = 1$ $(P(0) \wedge (P(0) \implies P(1))) \implies P(1)$

plus inductive step

Gauss and Induction

Child Gauss: $(\forall \mathbf{n} \in \mathbf{N})(\sum_{i=0}^{n} i = \frac{n(n+1)}{2})$ Proof?

Idea: assume predicate $P(n)$ for $n = k$. $P(k)$ is $\sum_{i=0}^{k} i = \frac{k(k+1)}{2}$.

Is predicate, $P(n)$ true for $n = k+1$?

$\sum_{i=0}^{k+1} i = (\sum_{i=1}^{k} i) + (k+1) = \frac{k(k+1)}{2} + k + 1 = \frac{k(k+1)+2(k+1)}{2} = \frac{(k+1)(k+2)}{2}$.

How about $k+2$. Same argument starting at $k+1$ works!

Induction Step. $P(k) \implies P(k+1)$.

Is this a proof? It shows that we can always move to the next step.

Need to start somewhere. $P(0)$ is $\sum_{i=0}^{0} i = \frac{(0)(0+1)}{2}$ **Base Case.**

Statement is true for $n = 0$ $P(0)$ is true
plus inductive step $\implies$ true for $n = 1$ $(P(0) \wedge (P(0) \implies P(1))) \implies P(1)$
plus inductive step $\implies$ true for $n = 2$

Gauss and Induction

Child Gauss: $(\forall \mathbf{n} \in \mathbf{N})(\sum_{i=0}^{n} i = \frac{n(n+1)}{2})$ Proof?

Idea: assume predicate $P(n)$ for $n = k$. $P(k)$ is $\sum_{i=0}^{k} i = \frac{k(k+1)}{2}$.

Is predicate, $P(n)$ true for $n = k+1$?

$$\sum_{i=0}^{k+1} i = (\sum_{i=1}^{k} i) + (k+1) = \frac{k(k+1)}{2} + k + 1 = \frac{k(k+1)+2(k+1)}{2} = \frac{(k+1)(k+2)}{2}.$$

How about $k+2$. Same argument starting at $k+1$ works!

Induction Step. $P(k) \implies P(k+1)$.

Is this a proof? It shows that we can always move to the next step.

Need to start somewhere. $P(0)$ is $\sum_{i=0}^{0} i = \frac{(0)(0+1)}{2}$ **Base Case.**

Statement is true for $n = 0$ $P(0)$ is true

plus inductive step $\implies$ true for $n = 1$ $(P(0) \wedge (P(0) \implies P(1))) \implies P(1)$

plus inductive step $\implies$ true for $n = 2$ $(P(1) \wedge (P(1) \implies P(2))) \implies P(2)$

Gauss and Induction

Child Gauss: $(\forall \mathbf{n} \in \mathbf{N})(\sum_{i=0}^{n} i = \frac{n(n+1)}{2})$ Proof?

Idea: assume predicate $P(n)$ for $n = k$. $P(k)$ is $\sum_{i=0}^{k} i = \frac{k(k+1)}{2}$.

Is predicate, $P(n)$ true for $n = k+1$?

$$\sum_{i=0}^{k+1} i = (\sum_{i=1}^{k} i) + (k+1) = \frac{k(k+1)}{2} + k + 1 = \frac{k(k+1)+2(k+1)}{2} = \frac{(k+1)(k+2)}{2}.$$

How about $k+2$. Same argument starting at $k+1$ works!

Induction Step. $P(k) \implies P(k+1)$.

Is this a proof? It shows that we can always move to the next step.

Need to start somewhere. $P(0)$ is $\sum_{i=0}^{0} i = \frac{(0)(0+1)}{2}$ **Base Case.**

Statement is true for $n = 0$ $P(0)$ is true

plus inductive step $\implies$ true for $n = 1$ $(P(0) \wedge (P(0) \implies P(1))) \implies P(1)$

plus inductive step $\implies$ true for $n = 2$ $(P(1) \wedge (P(1) \implies P(2))) \implies P(2)$

$\ldots$

Gauss and Induction

Child Gauss: $(\forall \mathbf{n} \in \mathbf{N})(\sum_{i=0}^{n} i = \frac{n(n+1)}{2})$ Proof?

Idea: assume predicate $P(n)$ for $n = k$. $P(k)$ is $\sum_{i=0}^{k} i = \frac{k(k+1)}{2}$.

Is predicate, $P(n)$ true for $n = k+1$?

$$\sum_{i=0}^{k+1} i = (\sum_{i=1}^{k} i) + (k+1) = \frac{k(k+1)}{2} + k + 1 = \frac{k(k+1)+2(k+1)}{2} = \frac{(k+1)(k+2)}{2}.$$

How about $k+2$. Same argument starting at $k+1$ works!

Induction Step. $P(k) \implies P(k+1)$.

Is this a proof? It shows that we can always move to the next step.

Need to start somewhere. $P(0)$ is $\sum_{i=0}^{0} i = \frac{(0)(0+1)}{2}$ **Base Case.**

Statement is true for $n = 0$ $P(0)$ is true

plus inductive step $\implies$ true for $n = 1$ $(P(0) \wedge (P(0) \implies P(1))) \implies P(1)$

plus inductive step $\implies$ true for $n = 2$ $(P(1) \wedge (P(1) \implies P(2))) \implies P(2)$

...

true for $n = k$

Gauss and Induction

Child Gauss: $(\forall \mathbf{n} \in \mathbf{N})(\sum_{i=0}^{n} i = \frac{n(n+1)}{2})$ Proof?

Idea: assume predicate $P(n)$ for $n = k$. $P(k)$ is $\sum_{i=0}^{k} i = \frac{k(k+1)}{2}$.

Is predicate, $P(n)$ true for $n = k+1$?

$$\sum_{i=0}^{k+1} i = (\sum_{i=1}^{k} i) + (k+1) = \frac{k(k+1)}{2} + k + 1 = \frac{k(k+1)+2(k+1)}{2} = \frac{(k+1)(k+2)}{2}.$$

How about $k+2$. Same argument starting at $k+1$ works!

Induction Step. $P(k) \implies P(k+1)$.

Is this a proof? It shows that we can always move to the next step.

Need to start somewhere. $P(0)$ is $\sum_{i=0}^{0} i = \frac{(0)(0+1)}{2}$ **Base Case.**

Statement is true for $n = 0$ $P(0)$ is true

plus inductive step $\implies$ true for $n = 1$ $(P(0) \wedge (P(0) \implies P(1))) \implies P(1)$

plus inductive step $\implies$ true for $n = 2$ $(P(1) \wedge (P(1) \implies P(2))) \implies P(2)$

$\ldots$

true for $n = k \implies$ true for $n = k+1$

Gauss and Induction

Child Gauss: $(\forall \mathbf{n} \in \mathbf{N})(\sum_{i=0}^{n} i = \frac{n(n+1)}{2})$ Proof?

Idea: assume predicate $P(n)$ for $n = k$. $P(k)$ is $\sum_{i=0}^{k} i = \frac{k(k+1)}{2}$.

Is predicate, $P(n)$ true for $n = k+1$?

$$\sum_{i=0}^{k+1} i = (\sum_{i=1}^{k} i) + (k+1) = \frac{k(k+1)}{2} + k + 1 = \frac{k(k+1)+2(k+1)}{2} = \frac{(k+1)(k+2)}{2}.$$

How about $k+2$. Same argument starting at $k+1$ works!

Induction Step. $P(k) \implies P(k+1)$.

Is this a proof? It shows that we can always move to the next step.

Need to start somewhere. $P(0)$ is $\sum_{i=0}^{0} i = \frac{(0)(0+1)}{2}$ **Base Case.**

Statement is true for $n = 0$ $P(0)$ is true

plus inductive step $\implies$ true for $n = 1$ $(P(0) \wedge (P(0) \implies P(1))) \implies P(1)$

plus inductive step $\implies$ true for $n = 2$ $(P(1) \wedge (P(1) \implies P(2))) \implies P(2)$

...

true for $n = k \implies$ true for $n = k+1$ $(P(k) \wedge (P(k) \implies P(k+1))) \implies P(k+1)$

Gauss and Induction

Child Gauss: $(\forall \mathbf{n} \in \mathbf{N})(\sum_{i=0}^{n} i = \frac{n(n+1)}{2})$ Proof?

Idea: assume predicate $P(n)$ for $n = k$. $P(k)$ is $\sum_{i=0}^{k} i = \frac{k(k+1)}{2}$.

Is predicate, $P(n)$ true for $n = k+1$?

$$\sum_{i=0}^{k+1} i = (\sum_{i=1}^{k} i) + (k+1) = \frac{k(k+1)}{2} + k + 1 = \frac{k(k+1)+2(k+1)}{2} = \frac{(k+1)(k+2)}{2}.$$

How about $k+2$. Same argument starting at $k+1$ works!

Induction Step. $P(k) \implies P(k+1)$.

Is this a proof? It shows that we can always move to the next step.

Need to start somewhere. $P(0)$ is $\sum_{i=0}^{0} i = \frac{(0)(0+1)}{2}$ **Base Case.**

Statement is true for $n = 0$ $P(0)$ is true

plus inductive step $\implies$ true for $n = 1$ $(P(0) \wedge (P(0) \implies P(1))) \implies P(1)$

plus inductive step $\implies$ true for $n = 2$ $(P(1) \wedge (P(1) \implies P(2))) \implies P(2)$

...

true for $n = k$ $\implies$ true for $n = k+1$ $(P(k) \wedge (P(k) \implies P(k+1))) \implies P(k+1)$

...

Gauss and Induction

Child Gauss: $(\forall \mathbf{n} \in \mathbf{N})(\sum_{i=0}^{n} i = \frac{n(n+1)}{2})$ Proof?

Idea: assume predicate $P(n)$ for $n = k$. $P(k)$ is $\sum_{i=0}^{k} i = \frac{k(k+1)}{2}$.

Is predicate, $P(n)$ true for $n = k+1$?

$$\sum_{i=0}^{k+1} i = (\sum_{i=1}^{k} i) + (k+1) = \frac{k(k+1)}{2} + k + 1 = \frac{k(k+1)+2(k+1)}{2} = \frac{(k+1)(k+2)}{2}.$$

How about $k+2$. Same argument starting at $k+1$ works!

Induction Step. $P(k) \implies P(k+1)$.

Is this a proof? It shows that we can always move to the next step.

Need to start somewhere. $P(0)$ is $\sum_{i=0}^{0} i = \frac{(0)(0+1)}{2}$ **Base Case.**

Statement is true for $n = 0$ $P(0)$ is true

plus inductive step $\implies$ true for $n = 1$ $(P(0) \wedge (P(0) \implies P(1))) \implies P(1)$

plus inductive step $\implies$ true for $n = 2$ $(P(1) \wedge (P(1) \implies P(2))) \implies P(2)$

...

true for $n = k \implies$ true for $n = k+1$ $(P(k) \wedge (P(k) \implies P(k+1))) \implies P(k+1)$

...

Gauss and Induction

Child Gauss: $(\forall \mathbf{n} \in \mathbf{N})(\sum_{i=0}^{n} i = \frac{n(n+1)}{2})$ Proof?

Idea: assume predicate $P(n)$ for $n = k$. $P(k)$ is $\sum_{i=0}^{k} i = \frac{k(k+1)}{2}$.

Is predicate, $P(n)$ true for $n = k+1$?

$$\sum_{i=0}^{k+1} i = (\sum_{i=1}^{k} i) + (k+1) = \frac{k(k+1)}{2} + k + 1 = \frac{k(k+1)+2(k+1)}{2} = \frac{(k+1)(k+2)}{2}.$$

How about $k+2$. Same argument starting at $k+1$ works!

Induction Step. $P(k) \implies P(k+1)$.

Is this a proof? It shows that we can always move to the next step.

Need to start somewhere. $P(0)$ is $\sum_{i=0}^{0} i = \frac{(0)(0+1)}{2}$ **Base Case.**

Statement is true for $n = 0$ $P(0)$ is true

plus inductive step $\implies$ true for $n = 1$ $(P(0) \wedge (P(0) \implies P(1))) \implies P(1)$

plus inductive step $\implies$ true for $n = 2$ $(P(1) \wedge (P(1) \implies P(2))) \implies P(2)$

...

true for $n = k$ $\implies$ true for $n = k+1$ $(P(k) \wedge (P(k) \implies P(k+1))) \implies P(k+1)$

...

Predicate, $P(n)$, True for all natural numbers!

Gauss and Induction

Child Gauss: $(\forall \mathbf{n} \in \mathbf{N})(\sum_{i=0}^{n} i = \frac{n(n+1)}{2})$ Proof?

Idea: assume predicate $P(n)$ for $n = k$. $P(k)$ is $\sum_{i=0}^{k} i = \frac{k(k+1)}{2}$.

Is predicate, $P(n)$ true for $n = k+1$?

$$\sum_{i=0}^{k+1} i = (\sum_{i=1}^{k} i) + (k+1) = \frac{k(k+1)}{2} + k + 1 = \frac{k(k+1)+2(k+1)}{2} = \frac{(k+1)(k+2)}{2}.$$

How about $k+2$. Same argument starting at $k+1$ works!

Induction Step. $P(k) \implies P(k+1)$.

Is this a proof? It shows that we can always move to the next step.

Need to start somewhere. $P(0)$ is $\sum_{i=0}^{0} i = \frac{(0)(0+1)}{2}$ **Base Case.**

Statement is true for $n = 0$ $P(0)$ is true

plus inductive step $\implies$ true for $n = 1$ $(P(0) \wedge (P(0) \implies P(1))) \implies P(1)$

plus inductive step $\implies$ true for $n = 2$ $(P(1) \wedge (P(1) \implies P(2))) \implies P(2)$

...

true for $n = k$ $\implies$ true for $n = k+1$ $(P(k) \wedge (P(k) \implies P(k+1))) \implies P(k+1)$

...

Predicate, $P(n)$, True for all natural numbers! **Proof by Induction.**

Poll: What did Gauss use in the proof?

(A) Every natural number has a next number.

(B) The recursive leap of faith.

(C) $2^k > k$.

(D) $\forall k \in \mathbb{N}, \frac{k(k+1)}{2} + k + 1 = \frac{(k+1)(k+2)}{2}$.

Another Induction Proof.

Theorem: For every $n \in N$, $n^3 - n$ is divisible by 3. $(3|(n^3 - n)$).

Another Induction Proof.

Theorem: For every $n \in N$, $n^3 - n$ is divisible by 3. $(3|(n^3 - n)$).

Proof:

Another Induction Proof.

Theorem: For every $n \in N$, $n^3 - n$ is divisible by 3. $(3|(n^3 - n)$).

Proof: By induction.

Another Induction Proof.

Theorem: For every $n \in N$, $n^3 - n$ is divisible by 3. $(3|(n^3 - n)$).

Proof: By induction.
Base Case: $P(0)$ is "$(0^3) - 0$" is divisible by 3.

Another Induction Proof.

Theorem: For every $n \in N$, $n^3 - n$ is divisible by 3. $(3|(n^3 - n)\ )$.

Proof: By induction.
Base Case: $P(0)$ is "$(0^3) - 0$" is divisible by 3. Yes!

Another Induction Proof.

Theorem: For every $n \in N$, $n^3 - n$ is divisible by 3. $(3|(n^3 - n)$).

Proof: By induction.
Base Case: $P(0)$ is "$(0^3) - 0$" is divisible by 3. Yes!
Induction Step: $(\forall k \in N), P(k) \implies P(k+1)$

Another Induction Proof.

Theorem: For every $n \in N$, $n^3 - n$ is divisible by 3. $(3|(n^3 - n)$).

Proof: By induction.
Base Case: $P(0)$ is "$(0^3) - 0$" is divisible by 3. Yes!
Induction Step: $(\forall k \in N), P(k) \implies P(k+1)$
Induction Hypothesis: $k^3 - k$ is divisible by 3.

Another Induction Proof.

Theorem: For every $n \in N$, $n^3 - n$ is divisible by 3. $(3|(n^3 - n))$.

Proof: By induction.
Base Case: $P(0)$ is "$(0^3) - 0$" is divisible by 3. Yes!
Induction Step: $(\forall k \in N), P(k) \implies P(k+1)$
Induction Hypothesis: $k^3 - k$ is divisible by 3.
 or $k^3 - k = 3q$ for some integer q.

Another Induction Proof.

Theorem: For every $n \in N$, $n^3 - n$ is divisible by 3. $(3|(n^3 - n))$.

Proof: By induction.
Base Case: $P(0)$ is "$(0^3) - 0$" is divisible by 3. Yes!
Induction Step: $(\forall k \in N), P(k) \implies P(k+1)$
Induction Hypothesis: $k^3 - k$ is divisible by 3.
or $k^3 - k = 3q$ for some integer q.

Another Induction Proof.

Theorem: For every $n \in N$, $n^3 - n$ is divisible by 3. ($3|(n^3 - n)$).

Proof: By induction.
Base Case: $P(0)$ is "$(0^3) - 0$" is divisible by 3. Yes!
Induction Step: $(\forall k \in N), P(k) \implies P(k+1)$
Induction Hypothesis: $k^3 - k$ is divisible by 3.
or $k^3 - k = 3q$ for some integer q.

$(k+1)^3 - (k+1)$

Another Induction Proof.

Theorem: For every $n \in N$, $n^3 - n$ is divisible by 3. $(3|(n^3 - n)$).

Proof: By induction.
Base Case: $P(0)$ is "$(0^3) - 0$" is divisible by 3. Yes!
Induction Step: $(\forall k \in N), P(k) \implies P(k+1)$
Induction Hypothesis: $k^3 - k$ is divisible by 3.
or $k^3 - k = 3q$ for some integer q.

$(k+1)^3 - (k+1) = k^3 + 3k^2 + 3k + 1 - (k+1)$

Another Induction Proof.

Theorem: For every $n \in N$, $n^3 - n$ is divisible by 3. $(3|(n^3 - n)$).

Proof: By induction.
Base Case: $P(0)$ is "$(0^3) - 0$" is divisible by 3. Yes!
Induction Step: $(\forall k \in N), P(k) \implies P(k+1)$
Induction Hypothesis: $k^3 - k$ is divisible by 3.
  or $k^3 - k = 3q$ for some integer q.

$$\begin{aligned}(k+1)^3 - (k+1) &= k^3 + 3k^2 + 3k + 1 - (k+1) \\ &= k^3 + 3k^2 + 2k\end{aligned}$$

Another Induction Proof.

Theorem: For every $n \in N$, $n^3 - n$ is divisible by 3. ($3 | (n^3 - n)$).

Proof: By induction.
Base Case: $P(0)$ is "$(0^3) - 0$" is divisible by 3. Yes!
Induction Step: $(\forall k \in N), P(k) \implies P(k+1)$
Induction Hypothesis: $k^3 - k$ is divisible by 3.
or $k^3 - k = 3q$ for some integer q.

$$\begin{aligned}(k+1)^3 - (k+1) &= k^3 + 3k^2 + 3k + 1 - (k+1) \\ &= k^3 + 3k^2 + 2k \\ &= (k^3 - k) + 3k^2 + 3k\end{aligned}$$

Another Induction Proof.

Theorem: For every $n \in N$, $n^3 - n$ is divisible by 3. $(3|(n^3 - n)$).

Proof: By induction.
Base Case: $P(0)$ is "$(0^3) - 0$" is divisible by 3. Yes!
Induction Step: $(\forall k \in N), P(k) \implies P(k+1)$
Induction Hypothesis: $k^3 - k$ is divisible by 3.
or $k^3 - k = 3q$ for some integer q.

$$\begin{aligned}(k+1)^3 - (k+1) &= k^3 + 3k^2 + 3k + 1 - (k+1) \\ &= k^3 + 3k^2 + 2k \\ &= (k^3 - k) + 3k^2 + 3k \quad \text{Subtract/add } k\end{aligned}$$

Another Induction Proof.

Theorem: For every $n \in N$, $n^3 - n$ is divisible by 3. $(3|(n^3 - n)\ )$.

Proof: By induction.
Base Case: $P(0)$ is "$(0^3) - 0$" is divisible by 3. Yes!
Induction Step: $(\forall k \in N), P(k) \implies P(k+1)$
Induction Hypothesis: $k^3 - k$ is divisible by 3.
or $k^3 - k = 3q$ for some integer q.

$$\begin{aligned}(k+1)^3 - (k+1) &= k^3 + 3k^2 + 3k + 1 - (k+1)\\ &= k^3 + 3k^2 + 2k\\ &= (k^3 - k) + 3k^2 + 3k \quad \text{Subtract/add } k\\ &= 3q + 3(k^2 + k)\end{aligned}$$

Another Induction Proof.

Theorem: For every $n \in N$, $n^3 - n$ is divisible by 3. ($3|(n^3 - n)$).

Proof: By induction.
Base Case: $P(0)$ is "$(0^3) - 0$" is divisible by 3. Yes!
Induction Step: $(\forall k \in N), P(k) \implies P(k+1)$
Induction Hypothesis: $k^3 - k$ is divisible by 3.
or $k^3 - k = 3q$ for some integer q.

$$\begin{aligned}(k+1)^3 - (k+1) &= k^3 + 3k^2 + 3k + 1 - (k+1) \\ &= k^3 + 3k^2 + 2k \\ &= (k^3 - k) + 3k^2 + 3k \quad \text{Subtract/add } k \\ &= 3q + 3(k^2 + k) \quad \text{Induction Hyp.}\end{aligned}$$

Another Induction Proof.

Theorem: For every $n \in N$, $n^3 - n$ is divisible by 3. $(3|(n^3 - n)$).

Proof: By induction.
Base Case: $P(0)$ is "$(0^3) - 0$" is divisible by 3. Yes!
Induction Step: $(\forall k \in N), P(k) \implies P(k+1)$
Induction Hypothesis: $k^3 - k$ is divisible by 3.
or $k^3 - k = 3q$ for some integer q.

$$\begin{aligned}(k+1)^3 - (k+1) &= k^3 + 3k^2 + 3k + 1 - (k+1) \\ &= k^3 + 3k^2 + 2k \\ &= (k^3 - k) + 3k^2 + 3k \quad \text{Subtract/add } k \\ &= 3q + 3(k^2 + k) \quad \text{Induction Hyp.} \quad \text{Factor.}\end{aligned}$$

Another Induction Proof.

Theorem: For every $n \in N$, $n^3 - n$ is divisible by 3. $(3|(n^3-n)\ )$.

Proof: By induction.
Base Case: $P(0)$ is "$(0^3) - 0$" is divisible by 3. Yes!
Induction Step: $(\forall k \in N), P(k) \implies P(k+1)$
Induction Hypothesis: $k^3 - k$ is divisible by 3.
or $k^3 - k = 3q$ for some integer q.

$$
\begin{aligned}
(k+1)^3 - (k+1) &= k^3 + 3k^2 + 3k + 1 - (k+1) \\
&= k^3 + 3k^2 + 2k \\
&= (k^3 - k) + 3k^2 + 3k \quad \text{Subtract/add } k \\
&= 3q + 3(k^2 + k) \qquad \text{Induction Hyp.} \quad \text{Factor.} \\
&= 3(q + k^2 + k)
\end{aligned}
$$

Another Induction Proof.

Theorem: For every $n \in N$, $n^3 - n$ is divisible by 3. $(3|(n^3 - n)$).

Proof: By induction.
Base Case: $P(0)$ is "$(0^3) - 0$" is divisible by 3. Yes!
Induction Step: $(\forall k \in N), P(k) \implies P(k+1)$
Induction Hypothesis: $k^3 - k$ is divisible by 3.
or $k^3 - k = 3q$ for some integer q.

$$\begin{aligned}(k+1)^3 - (k+1) &= k^3 + 3k^2 + 3k + 1 - (k+1) \\ &= k^3 + 3k^2 + 2k \\ &= (k^3 - k) + 3k^2 + 3k \quad \text{Subtract/add } k \\ &= 3q + 3(k^2 + k) \quad \text{Induction Hyp. Factor.} \\ &= 3(q + k^2 + k) \quad \text{(Un)Distributive + over} \times\end{aligned}$$

Another Induction Proof.

Theorem: For every $n \in N$, $n^3 - n$ is divisible by 3. $(3|(n^3 - n)$).

Proof: By induction.
Base Case: $P(0)$ is "$(0^3) - 0$" is divisible by 3. Yes!
Induction Step: $(\forall k \in N), P(k) \implies P(k+1)$
Induction Hypothesis: $k^3 - k$ is divisible by 3.
  or $k^3 - k = 3q$ for some integer q.

$$\begin{aligned}
(k+1)^3 - (k+1) &= k^3 + 3k^2 + 3k + 1 - (k+1) \\
&= k^3 + 3k^2 + 2k \\
&= (k^3 - k) + 3k^2 + 3k \quad \text{Subtract/add } k \\
&= 3q + 3(k^2 + k) \qquad \text{Induction Hyp. \quad Factor.} \\
&= 3(q + k^2 + k) \qquad \text{(Un)Distributive } + \text{ over } \times
\end{aligned}$$

Or $(k+1)^3 - (k+1) = 3(q + k^2 + k)$.

Another Induction Proof.

Theorem: For every $n \in N$, $n^3 - n$ is divisible by 3. $(3|(n^3 - n)$).

Proof: By induction.
Base Case: $P(0)$ is "$(0^3) - 0$" is divisible by 3. Yes!
Induction Step: $(\forall k \in N), P(k) \implies P(k+1)$
Induction Hypothesis: $k^3 - k$ is divisible by 3.
or $k^3 - k = 3q$ for some integer q.

$$
\begin{aligned}
(k+1)^3 - (k+1) &= k^3 + 3k^2 + 3k + 1 - (k+1) \\
&= k^3 + 3k^2 + 2k \\
&= (k^3 - k) + 3k^2 + 3k \quad \text{Subtract/add } k \\
&= 3q + 3(k^2 + k) \qquad \text{Induction Hyp. \quad Factor.} \\
&= 3(q + k^2 + k) \qquad \text{(Un)Distributive } + \text{ over } \times
\end{aligned}
$$

Or $(k+1)^3 - (k+1) = 3(q + k^2 + k)$.

$(q + k^2 + k)$ is integer (closed under addition and multiplication).

Another Induction Proof.

Theorem: For every $n \in N$, $n^3 - n$ is divisible by 3. $(3|(n^3 - n)$).

Proof: By induction.
Base Case: $P(0)$ is "$(0^3) - 0$" is divisible by 3. Yes!
Induction Step: $(\forall k \in N), P(k) \implies P(k+1)$
Induction Hypothesis: $k^3 - k$ is divisible by 3.
  or $k^3 - k = 3q$ for some integer q.

$$\begin{aligned}
(k+1)^3 - (k+1) &= k^3 + 3k^2 + 3k + 1 - (k+1) \\
&= k^3 + 3k^2 + 2k \\
&= (k^3 - k) + 3k^2 + 3k \quad \text{Subtract/add } k \\
&= 3q + 3(k^2 + k) \qquad \text{Induction Hyp. Factor.} \\
&= 3(q + k^2 + k) \qquad \text{(Un)Distributive } + \text{ over } \times
\end{aligned}$$

Or $(k+1)^3 - (k+1) = 3(q + k^2 + k)$.

$(q + k^2 + k)$ is integer (closed under addition and multiplication).
  $\implies (k+1)^3 - (k+1)$ is divisible by 3.

Another Induction Proof.

Theorem: For every $n \in N$, $n^3 - n$ is divisible by 3. $(3|(n^3 - n)$).

Proof: By induction.
Base Case: $P(0)$ is "$(0^3) - 0$" is divisible by 3. Yes!
Induction Step: $(\forall k \in N), P(k) \implies P(k+1)$
Induction Hypothesis: $k^3 - k$ is divisible by 3.
or $k^3 - k = 3q$ for some integer q.

$$
\begin{aligned}
(k+1)^3 - (k+1) &= k^3 + 3k^2 + 3k + 1 - (k+1) \\
&= k^3 + 3k^2 + 2k \\
&= (k^3 - k) + 3k^2 + 3k \quad \text{Subtract/add } k \\
&= 3q + 3(k^2 + k) \qquad \text{Induction Hyp.} \quad \text{Factor.} \\
&= 3(q + k^2 + k) \qquad \text{(Un)Distributive } + \text{ over } \times
\end{aligned}
$$

Or $(k+1)^3 - (k+1) = 3(q + k^2 + k)$.

$(q + k^2 + k)$ is integer (closed under addition and multiplication).
$\implies (k+1)^3 - (k+1)$ is divisible by 3.
Thus, $(\forall k \in N) P(k) \implies P(k+1)$

Another Induction Proof.

Theorem: For every $n \in N$, $n^3 - n$ is divisible by 3. $(3|(n^3 - n)$).

Proof: By induction.
Base Case: $P(0)$ is "$(0^3) - 0$" is divisible by 3. Yes!
Induction Step: $(\forall k \in N), P(k) \implies P(k+1)$
Induction Hypothesis: $k^3 - k$ is divisible by 3.
 or $k^3 - k = 3q$ for some integer q.

$$
\begin{aligned}
(k+1)^3 - (k+1) &= k^3 + 3k^2 + 3k + 1 - (k+1) \\
&= k^3 + 3k^2 + 2k \\
&= (k^3 - k) + 3k^2 + 3k \quad \text{Subtract/add } k \\
&= 3q + 3(k^2 + k) \qquad \text{Induction Hyp. \quad Factor.} \\
&= 3(q + k^2 + k) \qquad \text{(Un)Distributive + over } \times
\end{aligned}
$$

Or $(k+1)^3 - (k+1) = 3(q + k^2 + k)$.

$(q + k^2 + k)$ is integer (closed under addition and multiplication).
 $\implies (k+1)^3 - (k+1)$ is divisible by 3.
Thus, $(\forall k \in N) P(k) \implies P(k+1)$
Thus, theorem holds by induction.

Another Induction Proof.

Theorem: For every $n \in N$, $n^3 - n$ is divisible by 3. $(3|(n^3 - n)\ )$.

Proof: By induction.
Base Case: $P(0)$ is "$(0^3) - 0$" is divisible by 3. Yes!
Induction Step: $(\forall k \in N), P(k) \implies P(k+1)$
Induction Hypothesis: $k^3 - k$ is divisible by 3.
or $k^3 - k = 3q$ for some integer q.

$$
\begin{aligned}
(k+1)^3 - (k+1) &= k^3 + 3k^2 + 3k + 1 - (k+1) \\
&= k^3 + 3k^2 + 2k \\
&= (k^3 - k) + 3k^2 + 3k \quad \text{Subtract/add } k \\
&= 3q + 3(k^2 + k) \quad \text{Induction Hyp. Factor.} \\
&= 3(q + k^2 + k) \quad \text{(Un)Distributive } + \text{ over } \times
\end{aligned}
$$

Or $(k+1)^3 - (k+1) = 3(q + k^2 + k)$.

$(q + k^2 + k)$ is integer (closed under addition and multiplication).
$\implies (k+1)^3 - (k+1)$ is divisible by 3.
Thus, $(\forall k \in N) P(k) \implies P(k+1)$
Thus, theorem holds by induction. $\square$

Poll: steps of proof.

Theorem: For all natural numbers n, $3|n^3 - n$.

Poll: steps of proof.

Theorem: For all natural numbers n, $3|n^3 - n$.

What did we use in the proof?

Poll: steps of proof.

Theorem: For all natural numbers n, $3|n^3 - n$.

What did we use in the proof?

(A) $(\forall n \in \mathbb{N}, P(n) \implies P(n+1)) \implies (\forall n \in \mathbb{N}, P(n))$.

Poll: steps of proof.

Theorem: For all natural numbers n, $3|n^3 - n$.

What did we use in the proof?

(A) $(\forall n \in \mathbb{N}, P(n) \implies P(n+1)) \implies (\forall n \in \mathbb{N}, P(n))$.
(B) $\forall k \in \mathbb{N}, (3|k^3 - k) \implies (3|(k+1)^3 - (k+1))$.

Poll: steps of proof.

Theorem: For all natural numbers n, $3|n^3 - n$.

What did we use in the proof?

(A) $(\forall n \in \mathbb{N}, P(n) \implies P(n+1)) \implies (\forall n \in \mathbb{N}, P(n))$.
(B) $\forall k \in \mathbb{N}, (3|k^3 - k) \implies (3|(k+1)^3 - (k+1))$.
(C) $(k+1)^3 = k^3 + 3k^2 + 3k + 1$.

Poll: steps of proof.

Theorem: For all natural numbers n, $3|n^3 - n$.

What did we use in the proof?

(A) $(\forall n \in \mathbb{N}, P(n) \implies P(n+1)) \implies (\forall n \in \mathbb{N}, P(n))$.
(B) $\forall k \in \mathbb{N}, (3|k^3 - k) \implies (3|(k+1)^3 - (k+1))$.
(C) $(k+1)^3 = k^3 + 3k^2 + 3k + 1$.
(D) $(k+1)^3 - (k+1) = k^3 + 3k^2 + 2k$

Poll: steps of proof.

Theorem: For all natural numbers n, $3|n^3 - n$.

What did we use in the proof?

(A) $(\forall n \in \mathbb{N}, P(n) \implies P(n+1)) \implies (\forall n \in \mathbb{N}, P(n))$.
(B) $\forall k \in \mathbb{N}, (3|k^3 - k) \implies (3|(k+1)^3 - (k+1))$.
(C) $(k+1)^3 = k^3 + 3k^2 + 3k + 1$.
(D) $(k+1)^3 - (k+1) = k^3 + 3k^2 + 2k$
(E) $k^3 + 3k^2 + 2k = (k^3 - k) + 3k^2 + k$

Poll: steps of proof.

Theorem: For all natural numbers n, $3|n^3 - n$.

What did we use in the proof?

(A) $(\forall n \in \mathbb{N}, P(n) \implies P(n+1)) \implies (\forall n \in \mathbb{N}, P(n))$.
(B) $\forall k \in \mathbb{N}, (3|k^3 - k) \implies (3|(k+1)^3 - (k+1))$.
(C) $(k+1)^3 = k^3 + 3k^2 + 3k + 1$.
(D) $(k+1)^3 - (k+1) = k^3 + 3k^2 + 2k$
(E) $k^3 + 3k^2 + 2k = (k^3 - k) + 3k^2 + k$

Poll: steps of proof.

Theorem: For all natural numbers n, $3|n^3 - n$.

What did we use in the proof?

(A) $(\forall n \in \mathbb{N}, P(n) \implies P(n+1)) \implies (\forall n \in \mathbb{N}, P(n))$.
(B) $\forall k \in \mathbb{N}, (3|k^3 - k) \implies (3|(k+1)^3 - (k+1))$.
(C) $(k+1)^3 = k^3 + 3k^2 + 3k + 1$.
(D) $(k+1)^3 - (k+1) = k^3 + 3k^2 + 2k$
(E) $k^3 + 3k^2 + 2k = (k^3 - k) + 3k^2 + k$

We used everything above except (A) and (E), cuz is false.

Two color theorem: example.

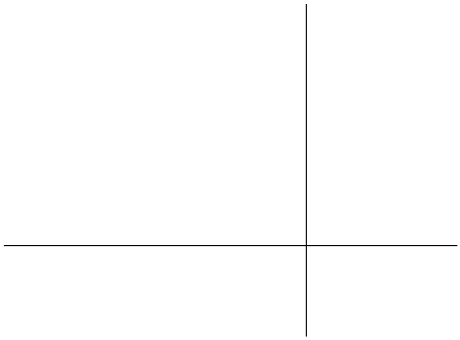
Any map formed by dividing the plane into regions by drawing straight lines can be properly colored with two colors.

.

Proper coloring: for each line segment the regions on the two sides have different colors.1

Two color theorem: example.

Any map formed by dividing the plane into regions by drawing straight lines can be properly colored with two colors.

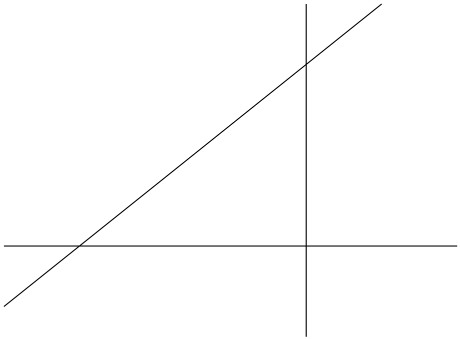


.

Proper coloring: for each line segment the regions on the two sides have different colors.1

Two color theorem: example.

Any map formed by dividing the plane into regions by drawing straight lines can be properly colored with two colors.

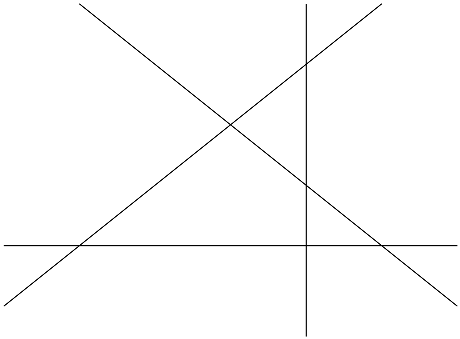


.

Proper coloring: for each line segment the regions on the two sides have different colors.1

Two color theorem: example.

Any map formed by dividing the plane into regions by drawing straight lines can be properly colored with two colors.

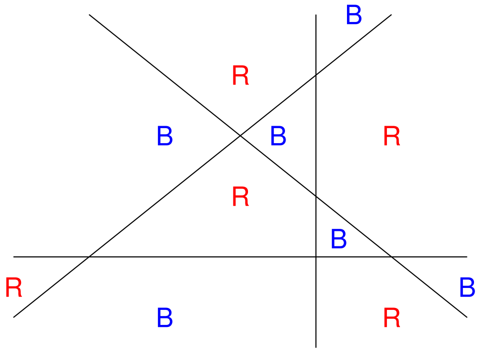


.

Proper coloring: for each line segment the regions on the two sides have different colors.1

Two color theorem: example.

Any map formed by dividing the plane into regions by drawing straight lines can be properly colored with two colors.

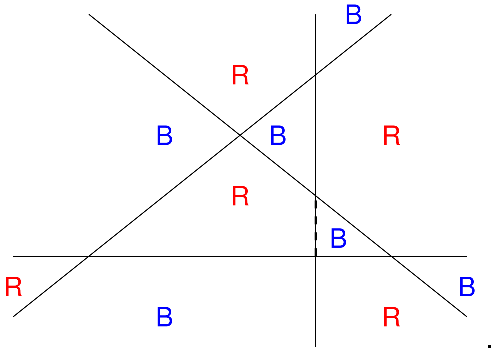


.

Proper coloring: for each line segment the regions on the two sides have different colors.1

Two color theorem: example.

Any map formed by dividing the plane into regions by drawing straight lines can be properly colored with two colors.

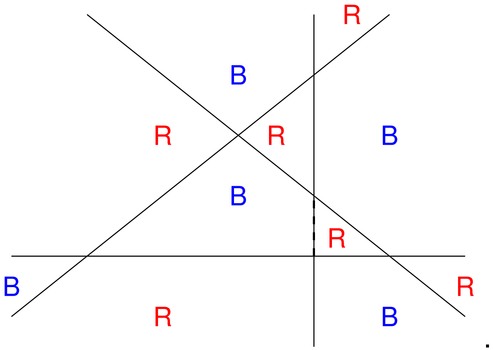


Proper coloring: for each line segment the regions on the two sides have different colors.1

Fact: Swapping red and blue gives another valid colors.

Two color theorem: example.

Any map formed by dividing the plane into regions by drawing straight lines can be properly colored with two colors.

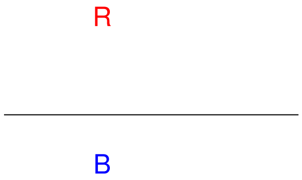


Proper coloring: for each line segment the regions on the two sides have different colors.1
Fact: Swapping red and blue gives another valid colors.

Two color theorem: proof illustration.

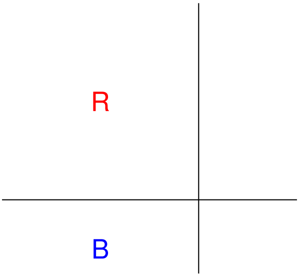
Base Case.

Two color theorem: proof illustration.



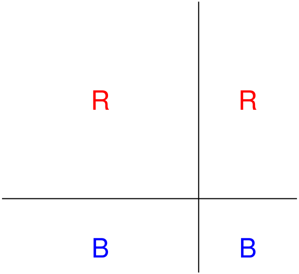
Base Case.

Two color theorem: proof illustration.



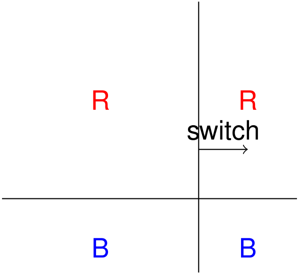
1. Add line.

Two color theorem: proof illustration.



1. Add line.
2. Get inherited color for split regions

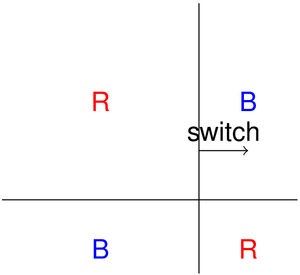
Two color theorem: proof illustration.



1. Add line.
2. Get inherited color for split regions
3. Switch on one side of new line.

(Fixes conflicts along new line, and makes no new ones along previous line.)

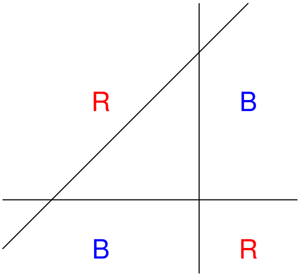
Two color theorem: proof illustration.



1. Add line.
2. Get inherited color for split regions
3. Switch on one side of new line.

(Fixes conflicts along new line, and makes no new ones along previous line.)

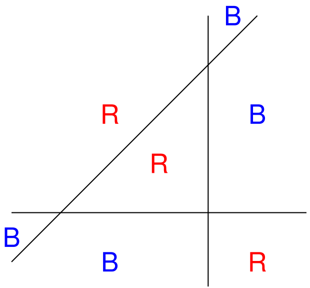
Two color theorem: proof illustration.



1. Add line.
2. Get inherited color for split regions
3. Switch on one side of new line.

(Fixes conflicts along new line, and makes no new ones along previous line.)

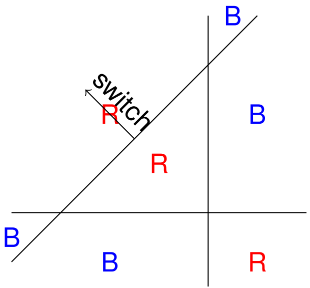
Two color theorem: proof illustration.



1. Add line.
2. Get inherited color for split regions
3. Switch on one side of new line.

(Fixes conflicts along new line, and makes no new ones along previous line.)

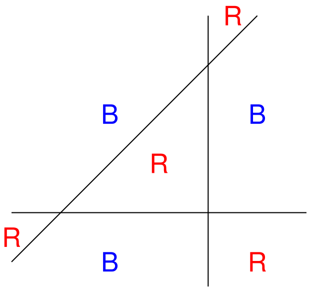
Two color theorem: proof illustration.



1. Add line.
2. Get inherited color for split regions
3. Switch on one side of new line.

(Fixes conflicts along new line, and makes no new ones along previous line.)

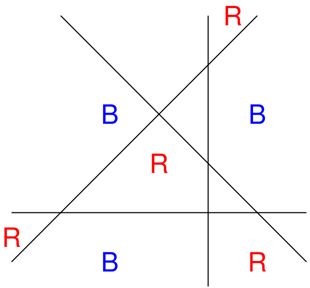
Two color theorem: proof illustration.



1. Add line.
2. Get inherited color for split regions
3. Switch on one side of new line.

(Fixes conflicts along new line, and makes no new ones along previous line.)

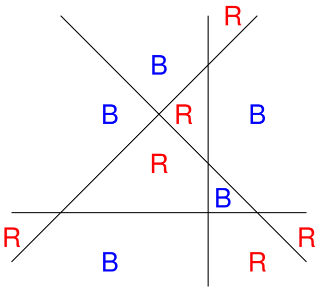
Two color theorem: proof illustration.



1. Add line.
2. Get inherited color for split regions
3. Switch on one side of new line.

(Fixes conflicts along new line, and makes no new ones along previous line.)

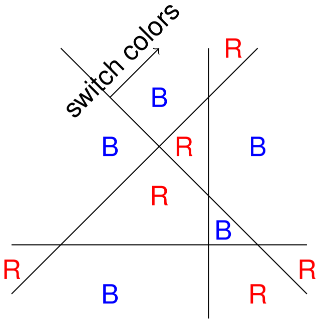
Two color theorem: proof illustration.



1. Add line.
2. Get inherited color for split regions
3. Switch on one side of new line.

(Fixes conflicts along new line, and makes no new ones along previous line.)

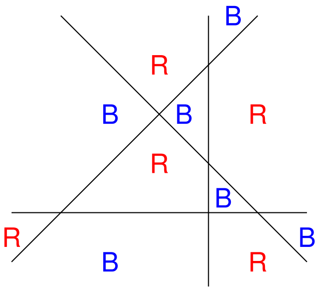
Two color theorem: proof illustration.



1. Add line.
2. Get inherited color for split regions
3. Switch on one side of new line.

(Fixes conflicts along new line, and makes no new ones along previous line.)

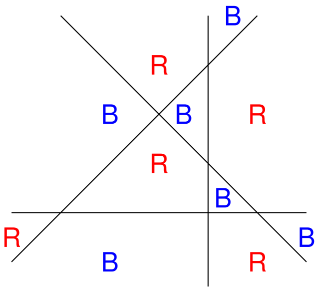
Two color theorem: proof illustration.



1. Add line.
2. Get inherited color for split regions
3. Switch on one side of new line.

(Fixes conflicts along new line, and makes no new ones along previous line.)

Two color theorem: proof illustration.

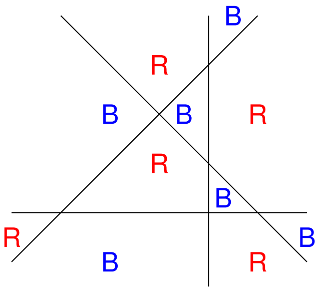


1. Add line.
2. Get inherited color for split regions
3. Switch on one side of new line.

(Fixes conflicts along new line, and makes no new ones along previous line.)

Algorithm gives $P(k) \implies P(k+1)$.

Two color theorem: proof illustration.



1. Add line.
2. Get inherited color for split regions
3. Switch on one side of new line.

(Fixes conflicts along new line, and makes no new ones along previous line.)

Algorithm gives $P(k) \implies P(k+1)$. $\square$

Poll: what did we use in the proof.

(A) Switching a 2-coloring is a valid coloring.

(B) Definition of 2-coloring.

(C) Definition of adjacent.

(D) Definition of region.

(E) The four color theorem.

Strenthening Induction Hypothesis.

Theorem: The sum of the first n odd numbers is a perfect square.

Strenthening Induction Hypothesis.

Theorem: The sum of the first n odd numbers is a perfect square.

kth odd number is $2(k-1)+1$.

Strenthening Induction Hypothesis.

Theorem: The sum of the first n odd numbers is a perfect square.

kth odd number is $2(k-1)+1$.

Base Case 1 (first odd number) is 1^2.

Strenthening Induction Hypothesis.

Theorem: The sum of the first n odd numbers is a perfect square.

kth odd number is $2(k-1)+1$.

Base Case 1 (first odd number) is 1^2.

Induction Hypothesis Sum of first k odds is perfect square a^2

Strenthening Induction Hypothesis.

Theorem: The sum of the first n odd numbers is a perfect square.

kth odd number is $2(k-1)+1$.

Base Case: 1 (first odd number) is 1^2.

Induction Hypothesis: Sum of first k odds is perfect square a^2

Induction Step:

1. The $(k+1)$st odd number is $2k+1$.

Strenthening Induction Hypothesis.

Theorem: The sum of the first n odd numbers is a perfect square.

kth odd number is $2(k-1)+1$.

Base Case: 1 (first odd number) is 1^2.

Induction Hypothesis: Sum of first k odds is perfect square a^2

Induction Step:

1. The $(k+1)$st odd number is $2k+1$.
2. Sum of the first $k+1$ odds is
   a^2+2k+1

Strenthening Induction Hypothesis.

Theorem: The sum of the first n odd numbers is a perfect square.

kth odd number is $2(k-1)+1$.

Base Case: 1 (first odd number) is 1^2.

Induction Hypothesis: Sum of first k odds is perfect square a^2

Induction Step:

1. The $(k+1)$st odd number is $2k+1$.
2. Sum of the first $k+1$ odds is
   a^2+2k+1
   ????

Strenthening Induction Hypothesis.

Theorem: The sum of the first n odd numbers is a perfect square.
Theorem: The sum of the first n odd numbers is n^2.

kth odd number is $2(k-1)+1$.

Base Case: 1 (first odd number) is 1^2.

Induction Hypothesis: Sum of first k odds is perfect square a^2

Induction Step:
1. The $(k+1)$st odd number is $2k+1$.
2. Sum of the first $k+1$ odds is a^2+2k+1

Strenthening Induction Hypothesis.

Theorem: The sum of the first n odd numbers is a perfect square.
Theorem: The sum of the first n odd numbers is n^2.

kth odd number is $2(k-1)+1$.

Base Case: 1 (first odd number) is 1^2.

Induction Hypothesis: Sum of first k odds is perfect square $a^2 = k^2$.

Induction Step:
1. The $(k+1)$st odd number is $2k+1$.
2. Sum of the first $k+1$ odds is
   $a^2 + 2k + 1$

Strenthening Induction Hypothesis.

Theorem: The sum of the first n odd numbers is a perfect square.
Theorem: The sum of the first n odd numbers is n^2.

kth odd number is $2(k-1)+1$.

Base Case 1 (first odd number) is 1^2.

Induction Hypothesis Sum of first k odds is perfect square $a^2 = k^2$.

Induction Step

1. The $(k+1)$st odd number is $2k+1$.
2. Sum of the first $k+1$ odds is
   $a^2 + 2k + 1 = k^2 + 2k + 1$

Strenthening Induction Hypothesis.

Theorem: The sum of the first n odd numbers is a perfect square.
Theorem: The sum of the first n odd numbers is n^2.

kth odd number is $2(k-1)+1$.

Base Case: 1 (first odd number) is 1^2.

Induction Hypothesis: Sum of first k odds is perfect square $a^2 = k^2$.

Induction Step:

1. The $(k+1)$st odd number is $2k+1$.
2. Sum of the first $k+1$ odds is
   $a^2 + 2k + 1 = k^2 + 2k + 1$
3. $k^2 + 2k + 1 = (k+1)^2$

Strenthening Induction Hypothesis.

Theorem: The sum of the first n odd numbers is a perfect square.
Theorem: The sum of the first n odd numbers is n^2.

kth odd number is $2(k-1)+1$.

Base Case: 1 (first odd number) is 1^2.

Induction Hypothesis: Sum of first k odds is perfect square $a^2 = k^2$.

Induction Step:

1. The $(k+1)$st odd number is $2k+1$.
2. Sum of the first $k+1$ odds is
   $a^2+2k+1 = k^2+2k+1$
3. $k^2+2k+1 = (k+1)^2$
   ... P(k+1)!

Strenthening Induction Hypothesis.

Theorem: The sum of the first n odd numbers is a perfect square.
Theorem: The sum of the first n odd numbers is n^2.

kth odd number is $2(k-1)+1$.

Base Case: 1 (first odd number) is 1^2.

Induction Hypothesis: Sum of first k odds is perfect square $a^2 = k^2$.

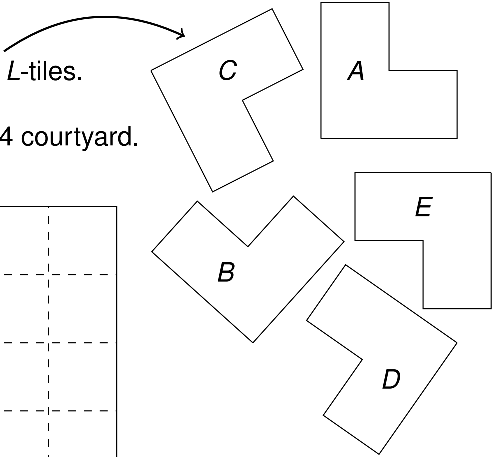
Induction Step:

1. The $(k+1)$st odd number is $2k+1$.
2. Sum of the first $k+1$ odds is
   $a^2+2k+1 = k^2+2k+1$
3. $k^2+2k+1 = (k+1)^2$
   ... P(k+1)!

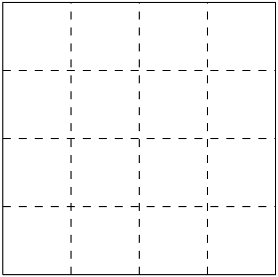
□

Tiling Cory Hall Courtyard.

Use these L-tiles.



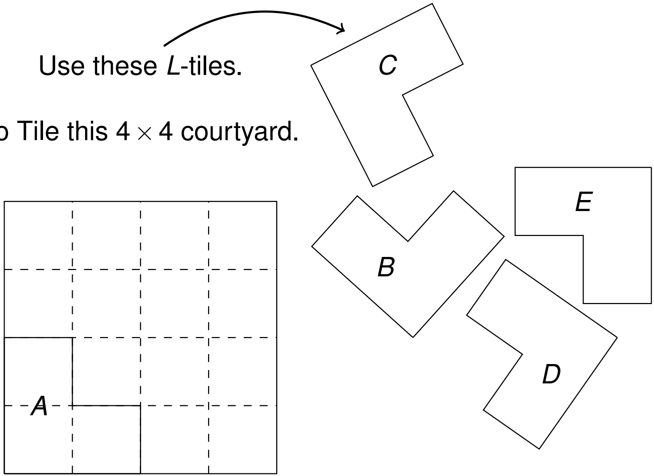
To Tile this 4×4 courtyard.



Tiling Cory Hall Courtyard.

Use these *L*-tiles.

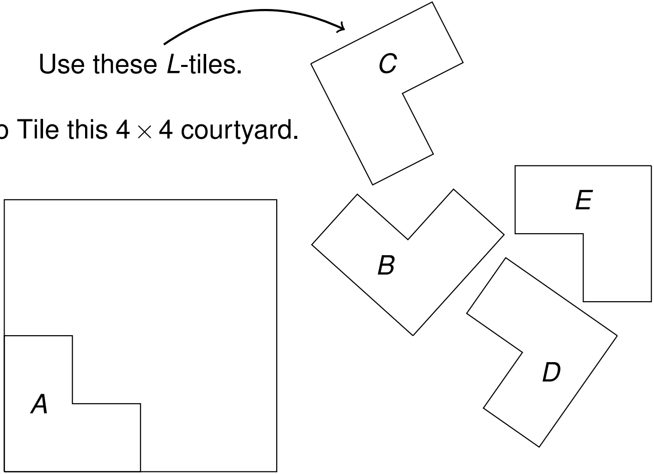
To Tile this 4×4 courtyard.



Tiling Cory Hall Courtyard.

Use these L-tiles.

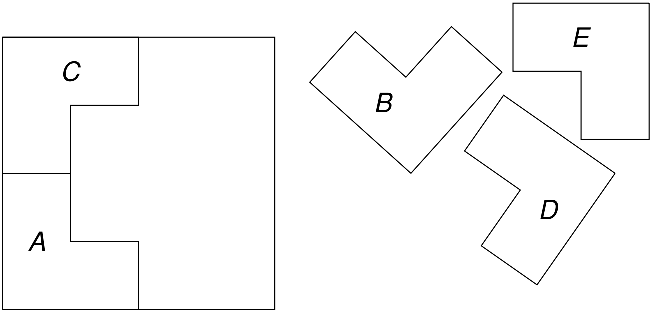
To Tile this 4×4 courtyard.



Tiling Cory Hall Courtyard.

Use these *L*-tiles.

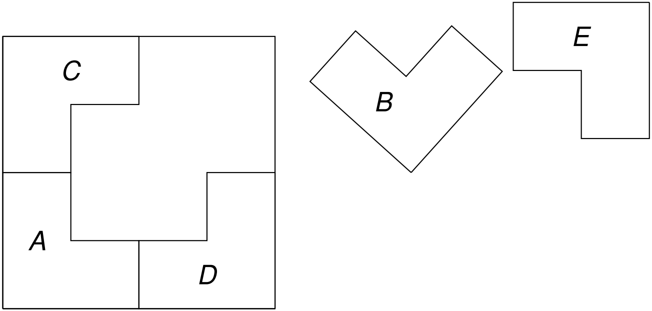
To Tile this 4×4 courtyard.



Tiling Cory Hall Courtyard.

Use these *L*-tiles.

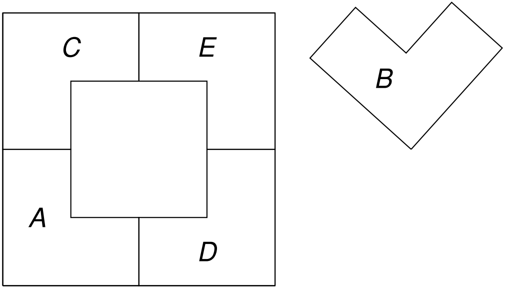
To Tile this 4×4 courtyard.



Tiling Cory Hall Courtyard.

Use these L-tiles.

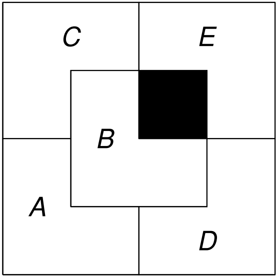
To Tile this 4×4 courtyard.



Tiling Cory Hall Courtyard.

Use these L-tiles.

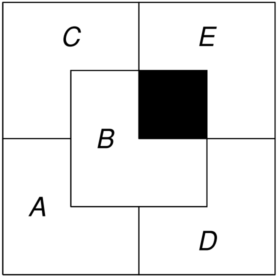
To Tile this 4×4 courtyard.



Tiling Cory Hall Courtyard.

Use these L-tiles.

To Tile this 4×4 courtyard.

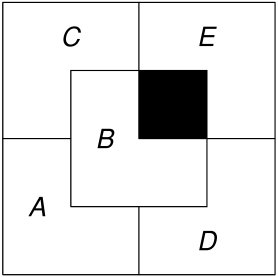


Alright!

Tiling Cory Hall Courtyard.

Use these L-tiles.

To Tile this 4×4 courtyard.



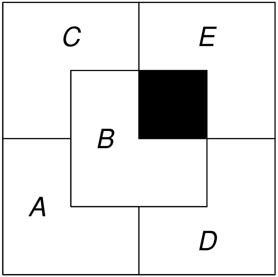
Alright!

Tiled 4×4 square with 2×2 L-tiles.

Tiling Cory Hall Courtyard.

Use these L-tiles.

To Tile this 4×4 courtyard.



Alright!

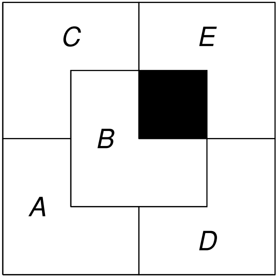
Tiled 4×4 square with 2×2 L-tiles.

with a center hole.

Tiling Cory Hall Courtyard.

Use these L-tiles.

To Tile this 4×4 courtyard.



Alright!

Tiled 4×4 square with 2×2 L-tiles.

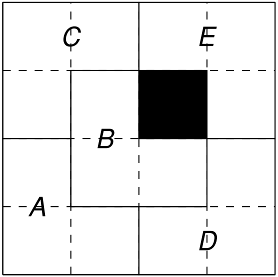
with a center hole.

Can we tile any $2^n \times 2^n$ with L-tiles (with a hole)

Tiling Cory Hall Courtyard.

Use these L-tiles.

To Tile this 4×4 courtyard.



Alright!

Tiled 4×4 square with 2×2 L-tiles.

with a center hole.

Can we tile any $2^n \times 2^n$ with L-tiles (with a hole) for every n!

Hole have to be there? Maybe just one?

Theorem: Any tiling of $2^n \times 2^n$ square has to have one hole.

Hole have to be there? Maybe just one?

Theorem: Any tiling of $2^n \times 2^n$ square has to have one hole.

Proof: The remainder of 2^{2n} divided by 3 is 1.

Hole have to be there? Maybe just one?

Theorem: Any tiling of $2^n \times 2^n$ square has to have one hole.

Proof: The remainder of 2^{2n} divided by 3 is 1.

Base case: true for $k = 0$.

Hole have to be there? Maybe just one?

Theorem: Any tiling of $2^n \times 2^n$ square has to have one hole.

Proof: The remainder of 2^{2n} divided by 3 is 1.

Base case: true for $k = 0$. $2^0 = 1$

Hole have to be there? Maybe just one?

Theorem: Any tiling of $2^n \times 2^n$ square has to have one hole.

Proof: The remainder of 2^{2n} divided by 3 is 1.

Base case: true for $k = 0$. $2^0 = 1$

Ind Hyp: $2^{2k} = 3a + 1$ for integer a.

Hole have to be there? Maybe just one?

Theorem: Any tiling of $2^n \times 2^n$ square has to have one hole.

Proof: The remainder of 2^{2n} divided by 3 is 1.

Base case: true for $k = 0$. $2^0 = 1$

Ind Hyp: $2^{2k} = 3a + 1$ for integer a.

$$2^{2(k+1)}$$

Hole have to be there? Maybe just one?

Theorem: Any tiling of $2^n \times 2^n$ square has to have one hole.

Proof: The remainder of 2^{2n} divided by 3 is 1.

Base case: true for $k = 0$. $2^0 = 1$

Ind Hyp: $2^{2k} = 3a + 1$ for integer a.

$$2^{2(k+1)} = 2^{2k} * 2^2$$

Hole have to be there? Maybe just one?

Theorem: Any tiling of $2^n \times 2^n$ square has to have one hole.

Proof: The remainder of 2^{2n} divided by 3 is 1.

Base case: true for $k = 0$. $2^0 = 1$

Ind Hyp: $2^{2k} = 3a + 1$ for integer a.

$$\begin{aligned} 2^{2(k+1)} &= 2^{2k} * 2^2 \\ &= 4 * 2^{2k} \end{aligned}$$

Hole have to be there? Maybe just one?

Theorem: Any tiling of $2^n \times 2^n$ square has to have one hole.

Proof: The remainder of 2^{2n} divided by 3 is 1.

Base case: true for $k = 0$. $2^0 = 1$

Ind Hyp: $2^{2k} = 3a + 1$ for integer a.

$$\begin{aligned} 2^{2(k+1)} &= 2^{2k} * 2^2 \\ &= 4 * 2^{2k} \\ &= 4 * (3a + 1) \end{aligned}$$

Hole have to be there? Maybe just one?

Theorem: Any tiling of $2^n \times 2^n$ square has to have one hole.

Proof: The remainder of 2^{2n} divided by 3 is 1.

Base case: true for $k = 0$. $2^0 = 1$

Ind Hyp: $2^{2k} = 3a + 1$ for integer a.

$$\begin{aligned} 2^{2(k+1)} &= 2^{2k} * 2^2 \\ &= 4 * 2^{2k} \\ &= 4 * (3a + 1) \\ &= 12a + 3 + 1 \end{aligned}$$

Hole have to be there? Maybe just one?

Theorem: Any tiling of $2^n \times 2^n$ square has to have one hole.

Proof: The remainder of 2^{2n} divided by 3 is 1.

Base case: true for $k = 0$. $2^0 = 1$

Ind Hyp: $2^{2k} = 3a + 1$ for integer a.

$$\begin{aligned}
2^{2(k+1)} &= 2^{2k} * 2^2 \\
&= 4 * 2^{2k} \\
&= 4 * (3a + 1) \\
&= 12a + 3 + 1 \\
&= 3(4a + 1) + 1
\end{aligned}$$

Hole have to be there? Maybe just one?

Theorem: Any tiling of $2^n \times 2^n$ square has to have one hole.

Proof: The remainder of 2^{2n} divided by 3 is 1.

Base case: true for $k = 0$. $2^0 = 1$

Ind Hyp: $2^{2k} = 3a + 1$ for integer a.

$$\begin{aligned}
2^{2(k+1)} &= 2^{2k} * 2^2 \\
&= 4 * 2^{2k} \\
&= 4 * (3a + 1) \\
&= 12a + 3 + 1 \\
&= 3(4a + 1) + 1
\end{aligned}$$

a integer

Hole have to be there? Maybe just one?

Theorem: Any tiling of $2^n \times 2^n$ square has to have one hole.

Proof: The remainder of 2^{2n} divided by 3 is 1.

Base case: true for $k = 0$. $2^0 = 1$

Ind Hyp: $2^{2k} = 3a + 1$ for integer a.

$$\begin{aligned} 2^{2(k+1)} &= 2^{2k} * 2^2 \\ &= 4 * 2^{2k} \\ &= 4 * (3a + 1) \\ &= 12a + 3 + 1 \\ &= 3(4a + 1) + 1 \end{aligned}$$

a integer $\implies (4a + 1)$ is an integer.

Hole have to be there? Maybe just one?

Theorem: Any tiling of $2^n \times 2^n$ square has to have one hole.

Proof: The remainder of 2^{2n} divided by 3 is 1.

Base case: true for $k = 0$. $2^0 = 1$

Ind Hyp: $2^{2k} = 3a + 1$ for integer a.

$$
\begin{aligned}
2^{2(k+1)} &= 2^{2k} * 2^2 \\
&= 4 * 2^{2k} \\
&= 4 * (3a + 1) \\
&= 12a + 3 + 1 \\
&= 3(4a + 1) + 1
\end{aligned}
$$

a integer $\implies (4a + 1)$ is an integer. $\square$

Hole in center?

Theorem: Can tile the $2^n \times 2^n$ square to leave a hole adjacent to the center.
Proof:

Hole in center?

Theorem: Can tile the $2^n \times 2^n$ square to leave a hole adjacent to the center.
Proof:

Base case: A single tile works fine.

Hole in center?

Theorem: Can tile the $2^n \times 2^n$ square to leave a hole adjacent to the center.

Proof:

Base case: A single tile works fine.

The hole is adjacent to the center of the 2×2 square.

Hole in center?

Theorem: Can tile the $2^n \times 2^n$ square to leave a hole adjacent to the center.
Proof:

Base case: A single tile works fine.
The hole is adjacent to the center of the 2×2 square.

Induction Hypothesis:

Hole in center?

Theorem: Can tile the $2^n \times 2^n$ square to leave a hole adjacent to the center.
Proof:

Base case: A single tile works fine.
The hole is adjacent to the center of the 2×2 square.

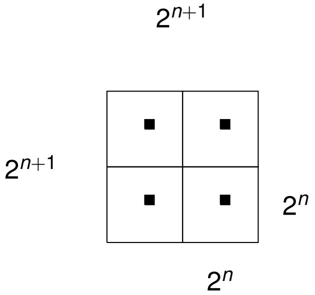
Induction Hypothesis:
Any $2^n \times 2^n$ square can be tiled with a hole at the center.

Hole in center?

Theorem: Can tile the $2^n \times 2^n$ square to leave a hole adjacent to the center.
Proof:

Base case: A single tile works fine.
The hole is adjacent to the center of the 2×2 square.

Induction Hypothesis:
Any $2^n \times 2^n$ square can be tiled with a hole at the center.

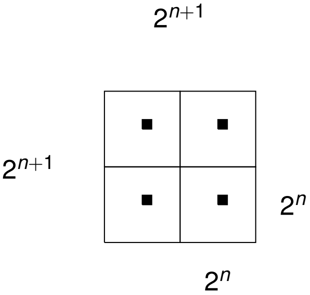


Hole in center?

Theorem: Can tile the $2^n \times 2^n$ square to leave a hole adjacent to the center.
Proof:

Base case: A single tile works fine.
The hole is adjacent to the center of the 2×2 square.

Induction Hypothesis:
Any $2^n \times 2^n$ square can be tiled with a hole at the center.



What to do now???

Hole can be anywhere!

Theorem: Can tile the $2^n \times 2^n$ to leave a hole adjacent *anywhere.*

Hole can be anywhere!

Theorem: Can tile the $2^n \times 2^n$ to leave a hole adjacent *anywhere.*

Better theorem

Hole can be anywhere!

Theorem: Can tile the $2^n \times 2^n$ to leave a hole adjacent *anywhere.*

Better theorem ...better induction hypothesis!

Hole can be anywhere!

Theorem: Can tile the $2^n \times 2^n$ to leave a hole adjacent *anywhere.*

Better theorem ...better induction hypothesis!

Base case: Sure. A tile is fine.

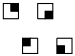
Hole can be anywhere!

Theorem: Can tile the $2^n \times 2^n$ to leave a hole adjacent *anywhere.*

Better theorem ...better induction hypothesis!

Base case: Sure. A tile is fine.

Flipping the orientation can leave hole anywhere.



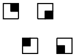
Hole can be anywhere!

Theorem: Can tile the $2^n \times 2^n$ to leave a hole adjacent *anywhere.*

Better theorem ...better induction hypothesis!

Base case: Sure. A tile is fine.

Flipping the orientation can leave hole anywhere.



Induction Hypothesis:

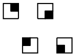
Hole can be anywhere!

Theorem: Can tile the $2^n \times 2^n$ to leave a hole adjacent *anywhere.*

Better theorem ...better induction hypothesis!

Base case: Sure. A tile is fine.

Flipping the orientation can leave hole anywhere.



Induction Hypothesis:
"Any $2^n \times 2^n$ square can be tiled with a hole **anywhere.**"
Consider $2^{n+1} \times 2^{n+1}$ square.

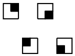
Hole can be anywhere!

Theorem: Can tile the $2^n \times 2^n$ to leave a hole adjacent *anywhere.*

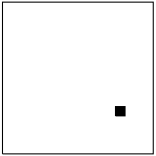
Better theorem ...better induction hypothesis!

Base case: Sure. A tile is fine.

Flipping the orientation can leave hole anywhere.



Induction Hypothesis:
"Any $2^n \times 2^n$ square can be tiled with a hole **anywhere.**"
Consider $2^{n+1} \times 2^{n+1}$ square.



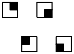
Hole can be anywhere!

Theorem: Can tile the $2^n \times 2^n$ to leave a hole adjacent *anywhere.*

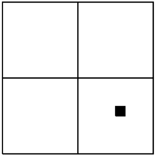
Better theorem ...better induction hypothesis!

Base case: Sure. A tile is fine.

Flipping the orientation can leave hole anywhere.



Induction Hypothesis:
"Any $2^n \times 2^n$ square can be tiled with a hole **anywhere.**"
Consider $2^{n+1} \times 2^{n+1}$ square.



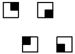
Use induction hypothesis in each.

Hole can be anywhere!

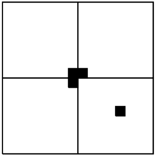
Theorem: Can tile the $2^n \times 2^n$ to leave a hole adjacent *anywhere.*

Better theorem ...better induction hypothesis!

Base case: Sure. A tile is fine.

Flipping the orientation can leave hole anywhere. 

Induction Hypothesis:
"Any $2^n \times 2^n$ square can be tiled with a hole **anywhere.**"
Consider $2^{n+1} \times 2^{n+1}$ square.



Use induction hypothesis in each.

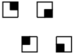
Hole can be anywhere!

Theorem: Can tile the $2^n \times 2^n$ to leave a hole adjacent *anywhere.*

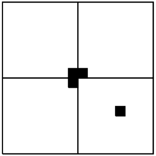
Better theorem ...better induction hypothesis!

Base case: Sure. A tile is fine.

Flipping the orientation can leave hole anywhere.



Induction Hypothesis:
"Any $2^n \times 2^n$ square can be tiled with a hole **anywhere.**"
Consider $2^{n+1} \times 2^{n+1}$ square.



Use induction hypothesis in each.

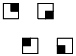
Use L-tile and ...

Hole can be anywhere!

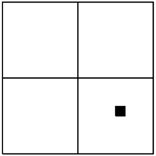
Theorem: Can tile the $2^n \times 2^n$ to leave a hole adjacent *anywhere.*

Better theorem ...better induction hypothesis!

Base case: Sure. A tile is fine.

Flipping the orientation can leave hole anywhere. 

Induction Hypothesis:
"Any $2^n \times 2^n$ square can be tiled with a hole **anywhere.**"
Consider $2^{n+1} \times 2^{n+1}$ square.



Use induction hypothesis in each.

Use L-tile and ... we are done.

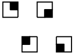
Hole can be anywhere!

Theorem: Can tile the $2^n \times 2^n$ to leave a hole adjacent *anywhere.*

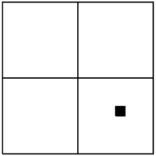
Better theorem ...better induction hypothesis!

Base case: Sure. A tile is fine.

Flipping the orientation can leave hole anywhere.



Induction Hypothesis:
"Any $2^n \times 2^n$ square can be tiled with a hole **anywhere.**"
Consider $2^{n+1} \times 2^{n+1}$ square.



Use induction hypothesis in each.

Use L-tile and ... we are done. □

Strong Induction.

Theorem: Every natural number $n > 1$ can be written as a (possibly trivial) product of primes.

Strong Induction.

Theorem: Every natural number $n > 1$ can be written as a (possibly trivial) product of primes.

Definition: A prime n has exactly 2 factors 1 and n.

Strong Induction.

Theorem: Every natural number $n > 1$ can be written as a (possibly trivial) product of primes.
Definition: A prime n has exactly 2 factors 1 and n.
Base Case: $n = 2$.

Strong Induction.

Theorem: Every natural number $n > 1$ can be written as a (possibly trivial) product of primes.

Definition: A prime n has exactly 2 factors 1 and n.

Base Case: $n = 2$.

Induction Step:

Strong Induction.

Theorem: Every natural number $n > 1$ can be written as a (possibly trivial) product of primes.

Definition: A prime n has exactly 2 factors 1 and n.

Base Case: $n = 2$.

Induction Step:

$P(n) =$ "n can be written as a product of primes. "

Strong Induction.

Theorem: Every natural number $n > 1$ can be written as a (possibly trivial) product of primes.
Definition: A prime n has exactly 2 factors 1 and n.
Base Case: $n = 2$.
Induction Step:
$P(n) =$ "n can be written as a product of primes. "
Either $n+1$ is a prime

Strong Induction.

Theorem: Every natural number $n > 1$ can be written as a (possibly trivial) product of primes.
Definition: A prime n has exactly 2 factors 1 and n.
Base Case: $n = 2$.
Induction Step:
$P(n) =$ "n can be written as a product of primes. "
Either $n+1$ is a prime or $n+1 = a \cdot b$ where $1 < a, b < n+1$.

Strong Induction.

Theorem: Every natural number $n > 1$ can be written as a (possibly trivial) product of primes.
Definition: A prime n has exactly 2 factors 1 and n.
Base Case: $n = 2$.
Induction Step:
$P(n) =$ "n can be written as a product of primes. "
Either $n+1$ is a prime or $n+1 = a \cdot b$ where $1 < a, b < n+1$.
$P(n)$ says nothing about a, b!

Strong Induction.

Theorem: Every natural number $n > 1$ can be written as a (possibly trivial) product of primes.
Definition: A prime n has exactly 2 factors 1 and n.
Base Case: $n = 2$.
Induction Step:
$P(n) =$ "n can be written as a product of primes. "
Either $n+1$ is a prime or $n+1 = a \cdot b$ where $1 < a, b < n+1$.
$P(n)$ says nothing about a, b!

Strong Induction Principle: If $P(0)$ and

$$(\forall k \in N)((P(0) \wedge \ldots \wedge P(k)) \implies P(k+1)),$$

then $(\forall k \in N)(P(k))$.

Strong Induction.

Theorem: Every natural number $n > 1$ can be written as a (possibly trivial) product of primes.
Definition: A prime n has exactly 2 factors 1 and n.
Base Case: $n = 2$.
Induction Step:
$P(n) =$ "n can be written as a product of primes. "
Either $n+1$ is a prime or $n+1 = a \cdot b$ where $1 < a, b < n+1$.
$P(n)$ says nothing about a, b!

Strong Induction Principle: If $P(0)$ and

$$(\forall k \in N)((P(0) \wedge \ldots \wedge P(k)) \implies P(k+1)),$$

then $(\forall k \in N)(P(k))$.

$$P(0) \implies P(1) \implies P(2) \implies P(3) \implies \cdots$$

Strong Induction.

Theorem: Every natural number $n > 1$ can be written as a (possibly trivial) product of primes.
Definition: A prime n has exactly 2 factors 1 and n.
Base Case: $n = 2$.
Induction Step:
$P(n) =$ "n can be written as a product of primes. "
Either $n+1$ is a prime or $n+1 = a \cdot b$ where $1 < a, b < n+1$.
$P(n)$ says nothing about a, b!

Strong Induction Principle: If $P(0)$ and

$$(\forall k \in N)((P(0) \wedge \ldots \wedge P(k)) \implies P(k+1)),$$

then $(\forall k \in N)(P(k))$.

$$P(0) \implies P(1) \implies P(2) \implies P(3) \implies \cdots$$

Strong Induction.

Theorem: Every natural number $n > 1$ can be written as a (possibly trivial) product of primes.
Definition: A prime n has exactly 2 factors 1 and n.
Base Case: $n = 2$.
Induction Step:
$P(n) =$ "n can be written as a product of primes. "
Either $n+1$ is a prime or $n+1 = a \cdot b$ where $1 < a, b < n+1$.
$P(n)$ says nothing about a, b!

Strong Induction Principle: If $P(0)$ and

$$(\forall k \in N)((P(0) \wedge \ldots \wedge P(k)) \implies P(k+1)),$$

then $(\forall k \in N)(P(k))$.

$$P(0) \implies P(1) \implies P(2) \implies P(3) \implies \cdots$$

Strong induction hypothesis: "a and b are products of primes"

Strong Induction.

Theorem: Every natural number $n > 1$ can be written as a (possibly trivial) product of primes.
Definition: A prime n has exactly 2 factors 1 and n.
Base Case: $n = 2$.
Induction Step:
$P(n) =$ "n can be written as a product of primes. "
Either $n+1$ is a prime or $n+1 = a \cdot b$ where $1 < a, b < n+1$.
$P(n)$ says nothing about a, b!

Strong Induction Principle: If $P(0)$ and

$$(\forall k \in N)((P(0) \wedge \ldots \wedge P(k)) \implies P(k+1)),$$

then $(\forall k \in N)(P(k))$.

$$P(0) \implies P(1) \implies P(2) \implies P(3) \implies \cdots$$

Strong induction hypothesis: "a and b are products of primes"

$\implies$ "$n+1 = a \cdot b$

Strong Induction.

Theorem: Every natural number $n > 1$ can be written as a (possibly trivial) product of primes.
Definition: A prime n has exactly 2 factors 1 and n.
Base Case: $n = 2$.
Induction Step:
$P(n) =$ "n can be written as a product of primes. "
Either $n+1$ is a prime or $n+1 = a \cdot b$ where $1 < a, b < n+1$.
$P(n)$ says nothing about a, b!

Strong Induction Principle: If $P(0)$ and

$$(\forall k \in N)((P(0) \wedge \ldots \wedge P(k)) \implies P(k+1)),$$

then $(\forall k \in N)(P(k))$.

$$P(0) \implies P(1) \implies P(2) \implies P(3) \implies \cdots$$

Strong induction hypothesis: "a and b are products of primes"

$\implies$ "$n+1 = a \cdot b =$ (factorization of a)

Strong Induction.

Theorem: Every natural number $n > 1$ can be written as a (possibly trivial) product of primes.
Definition: A prime n has exactly 2 factors 1 and n.
Base Case: $n = 2$.
Induction Step:
$P(n) =$ "n can be written as a product of primes. "
Either $n+1$ is a prime or $n+1 = a \cdot b$ where $1 < a, b < n+1$.
$P(n)$ says nothing about a, b!

Strong Induction Principle: If $P(0)$ and

$$(\forall k \in N)((P(0) \wedge \ldots \wedge P(k)) \implies P(k+1)),$$

then $(\forall k \in N)(P(k))$.

$$P(0) \implies P(1) \implies P(2) \implies P(3) \implies \cdots$$

Strong induction hypothesis: "a and b are products of primes"

$\implies$ "$n+1 = a \cdot b =$ (factorization of a)(factorization of b)"

Strong Induction.

Theorem: Every natural number $n > 1$ can be written as a (possibly trivial) product of primes.

Definition: A prime n has exactly 2 factors 1 and n.

Base Case: $n = 2$.

Induction Step:

$P(n) =$ "n can be written as a product of primes. "

Either $n+1$ is a prime or $n+1 = a \cdot b$ where $1 < a, b < n+1$.

$P(n)$ says nothing about a, b!

Strong Induction Principle: If $P(0)$ and

$$(\forall k \in N)((P(0) \wedge \ldots \wedge P(k)) \implies P(k+1)),$$

then $(\forall k \in N)(P(k))$.

$$P(0) \implies P(1) \implies P(2) \implies P(3) \implies \cdots$$

Strong induction hypothesis: "a and b are products of primes"

$\implies$ "$n+1 = a \cdot b =$ (factorization of a)(factorization of b)"

$n+1$ can be written as the product of the prime factors!

Strong Induction.

Theorem: Every natural number $n > 1$ can be written as a (possibly trivial) product of primes.
Definition: A prime n has exactly 2 factors 1 and n.
Base Case: $n = 2$.
Induction Step:
$P(n) =$ "n can be written as a product of primes. "
Either $n+1$ is a prime or $n+1 = a \cdot b$ where $1 < a, b < n+1$.
$P(n)$ says nothing about a, b!

Strong Induction Principle: If $P(0)$ and

$$(\forall k \in N)((P(0) \wedge \ldots \wedge P(k)) \implies P(k+1)),$$

then $(\forall k \in N)(P(k))$.

$$P(0) \implies P(1) \implies P(2) \implies P(3) \implies \cdots$$

Strong induction hypothesis: "a and b are products of primes"

$\implies$ "$n+1 = a \cdot b =$ (factorization of a)(factorization of b)"
$n+1$ can be written as the product of the prime factors!

□

Fix to last lecture.

A prime number is divisible by only itself and 1.

Fix to last lecture.

A prime number is divisible by only itself and 1.

A number that is not prime is divisible by another number.

Fix to last lecture.

A prime number is divisible by only itself and 1.

A number that is not prime is divisible by another number.

A number that is not prime is divisible by a prime.

Fix to last lecture.

A prime number is divisible by only itself and 1.

A number that is not prime is divisible by another number.

A number that is not prime is divisible by a prime.

Proof: induction.

Fix to last lecture.

A prime number is divisible by only itself and 1.

A number that is not prime is divisible by another number.

A number that is not prime is divisible by a prime.

Proof: induction.
  n is not prime.

Fix to last lecture.

A prime number is divisible by only itself and 1.

A number that is not prime is divisible by another number.

A number that is not prime is divisible by a prime.

Proof: induction.
n is not prime.
Divisible by another number, m, $n = jm$

Fix to last lecture.

A prime number is divisible by only itself and 1.

A number that is not prime is divisible by another number.

A number that is not prime is divisible by a prime.

Proof: induction.
n is not prime.
Divisible by another number, m, $n = jm$
Which is prime and we are done.

Fix to last lecture.

A prime number is divisible by only itself and 1.

A number that is not prime is divisible by another number.

A number that is not prime is divisible by a prime.

Proof: induction.
n is not prime.
Divisible by another number, m, $n = jm$
Which is prime and we are done.
Or $p|m$ for a prime p by induction hypothesis.

Fix to last lecture.

A prime number is divisible by only itself and 1.

A number that is not prime is divisible by another number.

A number that is not prime is divisible by a prime.

Proof: induction.
n is not prime.
Divisible by another number, m, $n = jm$
Which is prime and we are done.
Or $p|m$ for a prime p by induction hypothesis.
That is, $m = ip$ for integer i.

Fix to last lecture.

A prime number is divisible by only itself and 1.

A number that is not prime is divisible by another number.

A number that is not prime is divisible by a prime.

Proof: induction.
n is not prime.
Divisible by another number, m, $n = jm$
Which is prime and we are done.
Or $p|m$ for a prime p by induction hypothesis.
That is, $m = ip$ for integer i.
And $m|n$ or $n = jm$.

Fix to last lecture.

A prime number is divisible by only itself and 1.

A number that is not prime is divisible by another number.

A number that is not prime is divisible by a prime.

Proof: induction.
n is not prime.
Divisible by another number, m, $n = jm$
Which is prime and we are done.
Or $p|m$ for a prime p by induction hypothesis.
That is, $m = ip$ for integer i.
And $m|n$ or $n = jm$.
So $n = (ij)p$.

Fix to last lecture.

A prime number is divisible by only itself and 1.

A number that is not prime is divisible by another number.

A number that is not prime is divisible by a prime.

Proof: induction.
n is not prime.
Divisible by another number, m, $n = jm$
Which is prime and we are done.
Or $p|m$ for a prime p by induction hypothesis.
That is, $m = ip$ for integer i.
And $m|n$ or $n = jm$.
So $n = (ij)p$.

Prime p divides n by principle of strong induction. □

Well Ordering Principle and Induction.

If $(\forall n)P(n)$ is not true, then $(\exists n)\neg P(n)$.

Well Ordering Principle and Induction.

If $(\forall n)P(n)$ is not true, then $(\exists n)\neg P(n)$.

Consider smallest m, with $\neg P(m)$, $m \geq 0$

Well Ordering Principle and Induction.

If $(\forall n)P(n)$ is not true, then $(\exists n)\neg P(n)$.

Consider smallest m, with $\neg P(m)$, $m \geq 0$

$P(m-1) \implies P(m)$ must be false (assuming $P(0)$ holds.)

Well Ordering Principle and Induction.

If $(\forall n)P(n)$ is not true, then $(\exists n)\neg P(n)$.

Consider smallest m, with $\neg P(m)$, $m \geq 0$

$P(m-1) \implies P(m)$ must be false (assuming $P(0)$ holds.)

This is a restatement of the induction principle!
I.e.,

$$\neg(\forall n)P(n) \implies ((\exists n)\neg(P(n-1) \implies P(n)).$$

Well Ordering Principle and Induction.

If $(\forall n)P(n)$ is not true, then $(\exists n)\neg P(n)$.

Consider smallest m, with $\neg P(m)$, $m \geq 0$

$P(m-1) \implies P(m)$ must be false (assuming $P(0)$ holds.)

This is a restatement of the induction principle!
I.e.,

$$\neg(\forall n)P(n) \implies ((\exists n)\neg(P(n-1) \implies P(n)).$$

(Contrapositive of Induction principle (assuming $P(0)$))

Well Ordering Principle and Induction.

If $(\forall n)P(n)$ is not true, then $(\exists n)\neg P(n)$.

Consider smallest m, with $\neg P(m)$, $m \geq 0$

$P(m-1) \implies P(m)$ must be false (assuming $P(0)$ holds.)

This is a restatement of the induction principle!
I.e.,

$$\neg(\forall n)P(n) \implies ((\exists n)\neg(P(n-1) \implies P(n)).$$

(Contrapositive of Induction principle (assuming $P(0)$)

It assumes that there is a smallest m where $P(m)$ does not hold.

Well Ordering Principle and Induction.

If $(\forall n)P(n)$ is not true, then $(\exists n)\neg P(n)$.

Consider smallest m, with $\neg P(m)$, $m \geq 0$

$P(m-1) \implies P(m)$ must be false (assuming $P(0)$ holds.)

This is a restatement of the induction principle!
I.e.,

$$\neg(\forall n)P(n) \implies ((\exists n)\neg(P(n-1) \implies P(n)).$$

(Contrapositive of Induction principle (assuming $P(0)$)

It assumes that there is a smallest m where $P(m)$ does not hold.

The **Well ordering principle** states that for any subset of the natural numbers there is a smallest element.

Well Ordering Principle and Induction.

If $(\forall n)P(n)$ is not true, then $(\exists n)\neg P(n)$.

Consider smallest m, with $\neg P(m)$, $m \geq 0$

$P(m-1) \implies P(m)$ must be false (assuming $P(0)$ holds.)

This is a restatement of the induction principle!
I.e.,

$$\neg(\forall n)P(n) \implies ((\exists n)\neg(P(n-1) \implies P(n)).$$

(Contrapositive of Induction principle (assuming $P(0)$)

It assumes that there is a smallest m where $P(m)$ does not hold.

The **Well ordering principle** states that for any subset of the natural numbers there is a smallest element.

Examples: even numbers, odd numbers, primes, non-primes, etc..

Well Ordering Principle and Induction.

If $(\forall n)P(n)$ is not true, then $(\exists n)\neg P(n)$.

Consider smallest m, with $\neg P(m)$, $m \geq 0$

$P(m-1) \implies P(m)$ must be false (assuming $P(0)$ holds.)

This is a restatement of the induction principle!
I.e.,

$$\neg(\forall n)P(n) \implies ((\exists n)\neg(P(n-1) \implies P(n)).$$

(Contrapositive of Induction principle (assuming $P(0)$)

It assumes that there is a smallest m where $P(m)$ does not hold.

The **Well ordering principle** states that for any subset of the natural numbers there is a smallest element.

Examples: even numbers, odd numbers, primes, non-primes, etc..

True for rational numbers?

Well Ordering Principle and Induction.

If $(\forall n)P(n)$ is not true, then $(\exists n)\neg P(n)$.

Consider smallest m, with $\neg P(m)$, $m \geq 0$

$P(m-1) \implies P(m)$ must be false (assuming $P(0)$ holds.)

This is a restatement of the induction principle!
I.e.,

$$\neg(\forall n)P(n) \implies ((\exists n)\neg(P(n-1) \implies P(n)).$$

(Contrapositive of Induction principle (assuming $P(0)$)

It assumes that there is a smallest m where $P(m)$ does not hold.

The **Well ordering principle** states that for any subset of the natural numbers there is a smallest element.

Examples: even numbers, odd numbers, primes, non-primes, etc..

True for rational numbers? Poll.

Well Ordering Principle and Induction.

If $(\forall n)P(n)$ is not true, then $(\exists n)\neg P(n)$.

Consider smallest m, with $\neg P(m)$, $m \geq 0$

$P(m-1) \implies P(m)$ must be false (assuming $P(0)$ holds.)

This is a restatement of the induction principle!
I.e.,

$$\neg(\forall n)P(n) \implies ((\exists n)\neg(P(n-1) \implies P(n)).$$

(Contrapositive of Induction principle (assuming $P(0)$)

It assumes that there is a smallest m where $P(m)$ does not hold.

The **Well ordering principle** states that for any subset of the natural numbers there is a smallest element.

Examples: even numbers, odd numbers, primes, non-primes, etc..

True for rational numbers? Poll.

Note: can do with different definition of smallest.

Well Ordering Principle and Induction.

If $(\forall n)P(n)$ is not true, then $(\exists n)\neg P(n)$.

Consider smallest m, with $\neg P(m)$, $m \geq 0$

$P(m-1) \implies P(m)$ must be false (assuming $P(0)$ holds.)

This is a restatement of the induction principle!
I.e.,

$$\neg(\forall n)P(n) \implies ((\exists n)\neg(P(n-1) \implies P(n)).$$

(Contrapositive of Induction principle (assuming $P(0)$)

It assumes that there is a smallest m where $P(m)$ does not hold.

The **Well ordering principle** states that for any subset of the natural numbers there is a smallest element.

Examples: even numbers, odd numbers, primes, non-primes, etc..

True for rational numbers? Poll.

Note: can do with different definition of smallest.
For example. Use reduced form: a/b and order by $a+b$.

Well ordering principle.

Thm: All natural numbers are interesting.

Well ordering principle.

Thm: All natural numbers are interesting.

0 is interesting...

Well ordering principle.

Thm: All natural numbers are interesting.

0 is interesting...
Let *n* be the first uninteresting number.

Well ordering principle.

Thm: All natural numbers are interesting.

0 is interesting...
Let n be the first uninteresting number.
But $n-1$ is interesting and n is uninteresting,

Well ordering principle.

Thm: All natural numbers are interesting.

0 is interesting...
Let n be the first uninteresting number.
But $n-1$ is interesting and n is uninteresting,
so this is the first uninteresting number.

Well ordering principle.

Thm: All natural numbers are interesting.

0 is interesting...
Let n be the first uninteresting number.
But $n-1$ is interesting and n is uninteresting,
so this is the first uninteresting number.
But this is interesting.

Well ordering principle.

Thm: All natural numbers are interesting.

0 is interesting...
Let n be the first uninteresting number.
But $n-1$ is interesting and n is uninteresting,
so this is the first uninteresting number.
But this is interesting.
Thus, there is no smallest uninteresting natural number.

Well ordering principle.

Thm: All natural numbers are interesting.

0 is interesting...
Let n be the first uninteresting number.
But $n-1$ is interesting and n is uninteresting,
so this is the first uninteresting number.
But this is interesting.
Thus, there is no smallest uninteresting natural number.

Thus: All natural numbers are interesting.

Tournaments have short cycles

Def: A **round robin tournament on** n **players**: every player p plays every other player q, and either $p \rightarrow q$ (p beats q) or $q \rightarrow p$ (q beats p.)

Tournaments have short cycles

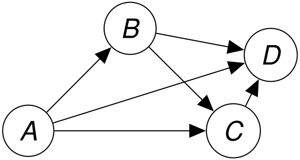
Def: A **round robin tournament on** n **players**: every player p plays every other player q, and either $p \to q$ (p beats q) or $q \to p$ (q beats p.)

Def: A **cycle**: a sequence of $p_1, \ldots, p_k$, $p_i \to p_{i+1}$ and $p_k \to p_1$.

Tournaments have short cycles

Def: A **round robin tournament on** n **players**: every player p plays every other player q, and either $p \rightarrow q$ (p beats q) or $q \rightarrow p$ (q beats p.)

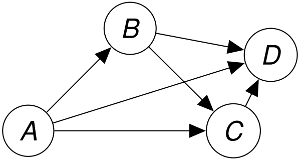
Def: A **cycle**: a sequence of $p_1, \ldots, p_k$, $p_i \rightarrow p_{i+1}$ and $p_k \rightarrow p_1$.



Tournaments have short cycles

Def: A **round robin tournament on n players**: every player p plays every other player q, and either $p \to q$ (p beats q) or $q \to p$ (q beats p.)

Def: A **cycle**: a sequence of $p_1, \ldots, p_k$, $p_i \to p_{i+1}$ and $p_k \to p_1$.

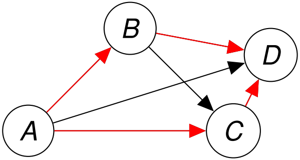


Theorem: Any tournament that has a cycle has a cycle of length 3.

Tournaments have short cycles

Def: A **round robin tournament on** n **players**: every player p plays every other player q, and either $p \rightarrow q$ (p beats q) or $q \rightarrow p$ (q beats p.)

Def: A **cycle**: a sequence of $p_1, \ldots, p_k$, $p_i \rightarrow p_{i+1}$ and $p_k \rightarrow p_1$.

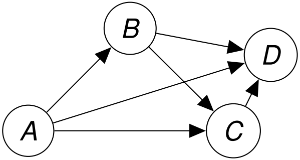


Theorem: Any tournament that has a cycle has a cycle of length 3.

Tournaments have short cycles

Def: A **round robin tournament on n players**: every player p plays every other player q, and either $p \rightarrow q$ (p beats q) or $q \rightarrow p$ (q beats p.)

Def: A **cycle**: a sequence of $p_1, \ldots, p_k$, $p_i \rightarrow p_{i+1}$ and $p_k \rightarrow p_1$.

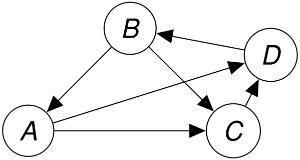


Theorem: Any tournament that has a cycle has a cycle of length 3.

Tournaments have short cycles

Def: A **round robin tournament on** n **players**: every player p plays every other player q, and either $p \rightarrow q$ (p beats q) or $q \rightarrow p$ (q beats p.)

Def: A **cycle**: a sequence of $p_1, \ldots, p_k$, $p_i \rightarrow p_{i+1}$ and $p_k \rightarrow p_1$.

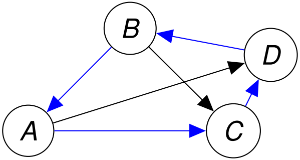


Theorem: Any tournament that has a cycle has a cycle of length 3.

Tournaments have short cycles

Def: A **round robin tournament on** n **players**: every player p plays every other player q, and either $p \to q$ (p beats q) or $q \to p$ (q beats p.)

Def: A **cycle**: a sequence of $p_1, \ldots, p_k$, $p_i \to p_{i+1}$ and $p_k \to p_1$.

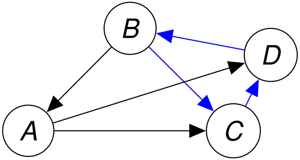


Theorem: Any tournament that has a cycle has a cycle of length 3.

Tournaments have short cycles

Def: A **round robin tournament on** n **players**: every player p plays every other player q, and either $p \rightarrow q$ (p beats q) or $q \rightarrow p$ (q beats p.)

Def: A **cycle**: a sequence of $p_1, \ldots, p_k$, $p_i \rightarrow p_{i+1}$ and $p_k \rightarrow p_1$.



Theorem: Any tournament that has a cycle has a cycle of length 3.

Tournament has a cycle of length 3 if at all.

Tournament has a cycle of length 3 if at all.

Assume the the **smallest cycle** is of length k.

Tournament has a cycle of length 3 if at all.

Assume the the **smallest cycle** is of length k.

Case 1: Of length 3.

Tournament has a cycle of length 3 if at all.

Assume the the **smallest cycle** is of length k.

Case 1: Of length 3. Done.

Tournament has a cycle of length 3 if at all.

Assume the the **smallest cycle** is of length k.

Case 1: Of length 3. Done.

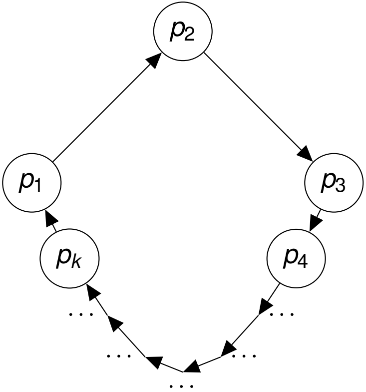
Case 2: Of length larger than 3.

Tournament has a cycle of length 3 if at all.

Assume the the **smallest cycle** is of length k.

Case 1: Of length 3. Done.

Case 2: Of length larger than 3.



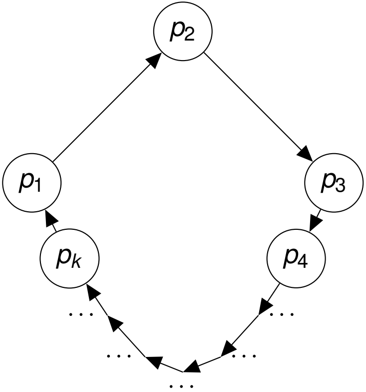
□

Tournament has a cycle of length 3 if at all.

Assume the the **smallest cycle** is of length k.

Case 1: Of length 3. Done.

Case 2: Of length larger than 3.



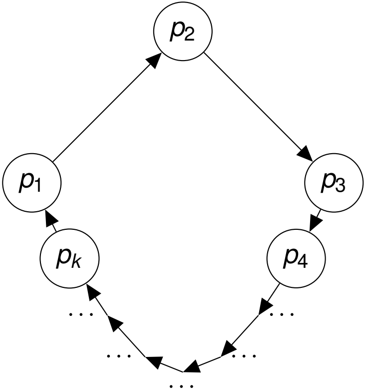
□

Tournament has a cycle of length 3 if at all.

Assume the the **smallest cycle** is of length k.

Case 1: Of length 3. Done.

Case 2: Of length larger than 3.

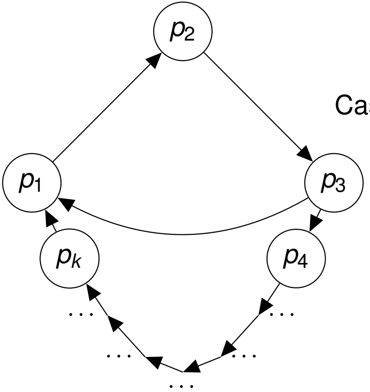


Tournament has a cycle of length 3 if at all.

Assume the the **smallest cycle** is of length k.

Case 1: Of length 3. Done.

Case 2: Of length larger than 3.



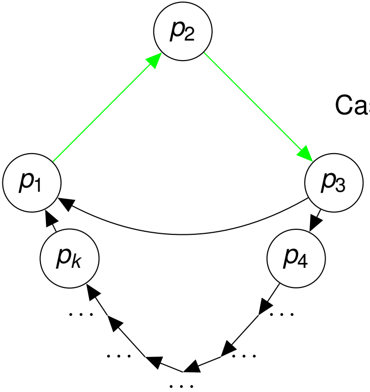
Case 2a: "$p_3 \rightarrow p_1$"

Tournament has a cycle of length 3 if at all.

Assume the the **smallest cycle** is of length k.

Case 1: Of length 3. Done.

Case 2: Of length larger than 3.



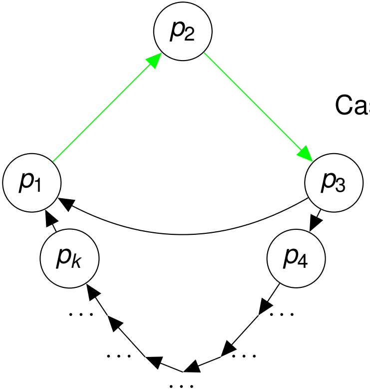
Case 2a: "$p_3 \rightarrow p_1$" $\Longrightarrow$ 3 cycle

Tournament has a cycle of length 3 if at all.

Assume the the **smallest cycle** is of length k.

Case 1: Of length 3. Done.

Case 2: Of length larger than 3.



Case 2a: "$p_3 \rightarrow p_1$" $\Longrightarrow$ 3 cycle

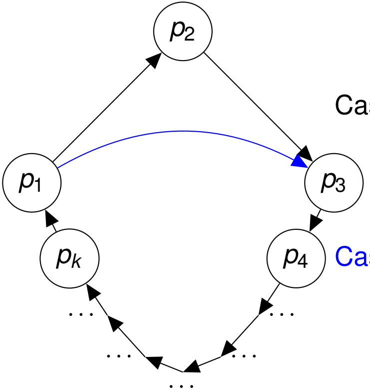
Contradiction.

Tournament has a cycle of length 3 if at all.

Assume the the **smallest cycle** is of length k.

Case 1: Of length 3. Done.

Case 2: Of length larger than 3.



Case 2a: "$p_3 \to p_1$" $\implies$ 3 cycle

Contradiction.

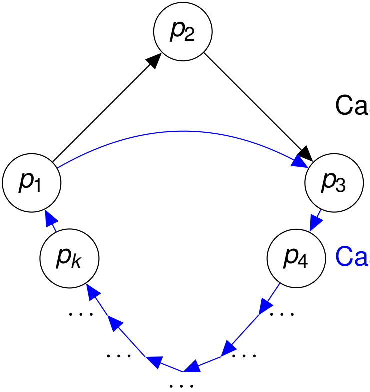
Case 2b: "$p_1 \to p_3$"

Tournament has a cycle of length 3 if at all.

Assume the the **smallest cycle** is of length k.

Case 1: Of length 3. Done.

Case 2: Of length larger than 3.



Case 2a: "$p_3 \rightarrow p_1$" $\implies$ 3 cycle

Contradiction.

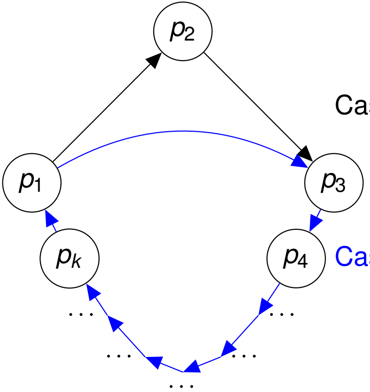
Case 2b: "$p_1 \rightarrow p_3$" $\implies$ $k-1$ length cycle!

Tournament has a cycle of length 3 if at all.

Assume the the **smallest cycle** is of length k.

Case 1: Of length 3. Done.

Case 2: Of length larger than 3.



Case 2a: "$p_3 \to p_1$" $\Longrightarrow$ 3 cycle

Contradiction.

Case 2b: "$p_1 \to p_3$" $\Longrightarrow$ $k-1$ length cycle!

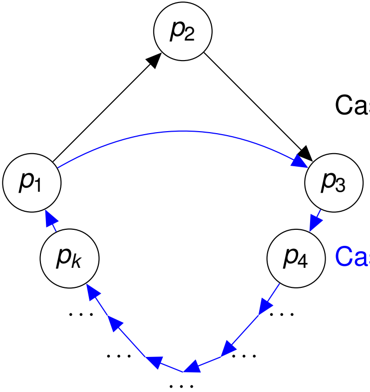
Contradicts assumption of smallest k!

Tournament has a cycle of length 3 if at all.

Assume the the **smallest cycle** is of length k.

Case 1: Of length 3. Done.

Case 2: Of length larger than 3.



Case 2a: "$p_3 \to p_1$" $\Longrightarrow$ 3 cycle

Contradiction.

Case 2b: "$p_1 \to p_3$" $\Longrightarrow$ $k-1$ length cycle!

Contradicts assumption of smallest k!

$\square$

Horses of the same color...

Theorem: All horses have the same color.

Horses of the same color...

Theorem: All horses have the same color.

Base Case: $P(1)$ - trivially true.

Horses of the same color...

Theorem: All horses have the same color.

Base Case: $P(1)$ - trivially true.

Induction Hypothesis: $P(k)$ - Any k horses have the same color.

Horses of the same color...

Theorem: All horses have the same color.

Base Case: $P(1)$ - trivially true.

Induction Hypothesis: $P(k)$ - Any k horses have the same color.

Induction step $P(k+1)$?

Horses of the same color...

Theorem: All horses have the same color.

Base Case: $P(1)$ - trivially true.

Induction Hypothesis: $P(k)$ - Any k horses have the same color.

Induction step $P(k+1)$?
First k have same color by $P(k)$. $1, 2, 3, \ldots, k, k+1$

Horses of the same color...

Theorem: All horses have the same color.

Base Case: $P(1)$ - trivially true.

Induction Hypothesis: $P(k)$ - Any k horses have the same color.

Induction step $P(k+1)$?
First k have same color by $P(k)$. $1,2,3,\ldots,k,k+1$
Second k have same color by $P(k)$. $1,2,3,\ldots,k,k+1$

Horses of the same color...

Theorem: All horses have the same color.

Base Case: $P(1)$ - trivially true.

Induction Hypothesis: $P(k)$ - Any k horses have the same color.

Induction step $P(k+1)$?

First k have same color by $P(k)$.	$1,2,3,\ldots,k,k+1$
Second k have same color by $P(k)$.	$1,2,3,\ldots,k,k+1$
A horse in the middle in common!	$1,2,3,\ldots,k,k+1$

Horses of the same color...

Theorem: All horses have the same color.

Base Case: $P(1)$ - trivially true.

Induction Hypothesis: $P(k)$ - Any k horses have the same color.

Induction step $P(k+1)$?

First k have same color by $P(k)$. $1, 2, 3, \ldots, k, k+1$

Second k have same color by $P(k)$. $1, 2, 3, \ldots, k, k+1$

A horse in the middle in common! $1, 2, 3, \ldots, k, k+1$

All k must have the same color. $1, 2, 3, \ldots, k, k+1$

Horses of the same color...

Theorem: All horses have the same color.

Base Case: $P(1)$ - trivially true.

Induction Hypothesis: $P(k)$ - Any k horses have the same color.

Induction step $P(k+1)$?
First k have same color by $P(k)$.
Second k have same color by $P(k)$.
A horse in the middle in common!

How about $P(1) \implies P(2)$?

Horses of the same color...

Theorem: All horses have the same color.

Base Case: $P(1)$ - trivially true.

Induction Hypothesis: $P(k)$ - Any k horses have the same color.

Induction step $P(k+1)$?
First k have same color by $P(k)$.
Second k have same color by $P(k)$.
A horse in the middle in common!

How about $P(1) \implies P(2)$?

Horses of the same color...

Theorem: All horses have the same color.

Base Case: $P(1)$ - trivially true.

Induction Hypothesis: $P(k)$ - Any k horses have the same color.

Induction step $P(k+1)$?
First k have same color by $P(k)$. $1,2$
Second k have same color by $P(k)$.
A horse in the middle in common!

How about $P(1) \implies P(2)$?

Horses of the same color...

Theorem: All horses have the same color.

Base Case: $P(1)$ - trivially true.

Induction Hypothesis: $P(k)$ - Any k horses have the same color.

Induction step $P(k+1)$?
First k have same color by $P(k)$. $1, 2$
Second k have same color by $P(k)$. $1, 2$
A horse in the middle in common!

How about $P(1) \implies P(2)$?

Horses of the same color...

Theorem: All horses have the same color.

Base Case: $P(1)$ - trivially true.

Induction Hypothesis: $P(k)$ - Any k horses have the same color.

Induction step $P(k+1)$?

First k have same color by $P(k)$. $1, 2$

Second k have same color by $P(k)$. $1, 2$

A horse in the middle in common! $1, 2$

No horse in common!

How about $P(1) \implies P(2)$?

Horses of the same color...

Theorem: All horses have the same color.

Base Case: $P(1)$ - trivially true.

Induction Hypothesis: $P(k)$ - Any k horses have the same color.

Induction step $P(k+1)$?

First k have same color by $P(k)$. $1, 2$

Second k have same color by $P(k)$. $1, 2$

A horse in the middle in common! $1, 2$

No horse in common!

How about $P(1) \implies P(2)$?

Horses of the same color...

Theorem: All horses have the same color.

Base Case: $P(1)$ - trivially true.
New Base Case: $P(2)$: there are two horses with same color.

Induction Hypothesis: $P(k)$ - Any k horses have the same color.

Induction step $P(k+1)$?
First k have same color by $P(k)$.
Second k have same color by $P(k)$.
A horse in the middle in common!

Fix base case.

Horses of the same color...

Theorem: All horses have the same color.

Base Case: $P(1)$ - trivially true.
New Base Case: $P(2)$: there are two horses with same color.

Induction Hypothesis: $P(k)$ - Any k horses have the same color.

Induction step $P(k+1)$?
First k have same color by $P(k)$.
Second k have same color by $P(k)$.
A horse in the middle in common!

Fix base case.
There are two horses of the same color.

Horses of the same color...

Theorem: All horses have the same color.

Base Case: $P(1)$ - trivially true.
New Base Case: $P(2)$: there are two horses with same color.

Induction Hypothesis: $P(k)$ - Any k horses have the same color.

Induction step $P(k+1)$?
First k have same color by $P(k)$.
Second k have same color by $P(k)$.
A horse in the middle in common!

Fix base case.
There are two horses of the same color. ...Still doesn't work!!

Horses of the same color...

Theorem: All horses have the same color.

Base Case: $P(1)$ - trivially true.
New Base Case: $P(2)$: there are two horses with same color.

Induction Hypothesis: $P(k)$ - Any k horses have the same color.

Induction step $P(k+1)$?
First k have same color by $P(k)$.
Second k have same color by $P(k)$.
A horse in the middle in common!

Fix base case.
There are two horses of the same color. ...Still doesn't work!!
(There are two horses is $\neq$ For every pair of two horses!!!)

Horses of the same color...

Theorem: All horses have the same color.

Base Case: $P(1)$ - trivially true.
New Base Case: $P(2)$: there are two horses with same color.

Induction Hypothesis: $P(k)$ - Any k horses have the same color.

Induction step $P(k+1)$?
First k have same color by $P(k)$.
Second k have same color by $P(k)$.
A horse in the middle in common!

Fix base case.
There are two horses of the same color. ...Still doesn't work!!
(There are two horses is $\neq$ For every pair of two horses!!!)

Of course it doesn't work.

Horses of the same color...

Theorem: All horses have the same color.

Base Case: $P(1)$ - trivially true.
New Base Case: $P(2)$: there are two horses with same color.

Induction Hypothesis: $P(k)$ - Any k horses have the same color.

Induction step $P(k+1)$?
First k have same color by $P(k)$.
Second k have same color by $P(k)$.
A horse in the middle in common!

Fix base case.
There are two horses of the same color. ...Still doesn't work!!
(There are two horses is $\neq$ For every pair of two horses!!!)

Of course it doesn't work.

More subtle to catch errors in proofs of correct theorems!!

Sad Islanders...

Island with 100 possibly blue-eyed and green-eyed inhabitants.

Sad Islanders...

Island with 100 possibly blue-eyed and green-eyed inhabitants.

Any islander who knows they have green eyes must “leave the island” that day.

Sad Islanders...

Island with 100 possibly blue-eyed and green-eyed inhabitants.

Any islander who knows they have green eyes must “leave the island” that day.

No islander knows there own eye color, but knows everyone elses.

Sad Islanders...

Island with 100 possibly blue-eyed and green-eyed inhabitants.

Any islander who knows they have green eyes must “leave the island” that day.

No islander knows there own eye color, but knows everyone elses.

All islanders have green eyes!

Sad Islanders...

Island with 100 possibly blue-eyed and green-eyed inhabitants.

Any islander who knows they have green eyes must “leave the island” that day.

No islander knows there own eye color, but knows everyone elses.

All islanders have green eyes!

First rule of island:

Sad Islanders...

Island with 100 possibly blue-eyed and green-eyed inhabitants.

Any islander who knows they have green eyes must “leave the island” that day.

No islander knows there own eye color, but knows everyone elses.

All islanders have green eyes!

First rule of island: Don’t talk about eye color!

Sad Islanders...

Island with 100 possibly blue-eyed and green-eyed inhabitants.

Any islander who knows they have green eyes must “leave the island” that day.

No islander knows there own eye color, but knows everyone elses.

All islanders have green eyes!

First rule of island: Don’t talk about eye color!

Visitor: “I see someone has green eyes.”

Sad Islanders...

Island with 100 possibly blue-eyed and green-eyed inhabitants.

Any islander who knows they have green eyes must “leave the island” that day.

No islander knows there own eye color, but knows everyone elses.

All islanders have green eyes!

First rule of island: Don’t talk about eye color!

Visitor: “I see someone has green eyes.”

Result: What happens?
(A) Nothing, no information was added.
(B) Information was added, maybe?
(C) They all leave the island.
(D) They all leave the island on day 100.

Sad Islanders...

Island with 100 possibly blue-eyed and green-eyed inhabitants.

Any islander who knows they have green eyes must “leave the island” that day.

No islander knows there own eye color, but knows everyone elses.

All islanders have green eyes!

First rule of island: Don’t talk about eye color!

Visitor: “I see someone has green eyes.”

Result: What happens?
(A) Nothing, no information was added.
(B) Information was added, maybe?
(C) They all leave the island.
(D) They all leave the island on day 100.

On day 100,

Sad Islanders...

Island with 100 possibly blue-eyed and green-eyed inhabitants.

Any islander who knows they have green eyes must “leave the island” that day.

No islander knows there own eye color, but knows everyone elses.

All islanders have green eyes!

First rule of island: Don’t talk about eye color!

Visitor: “I see someone has green eyes.”

Result: What happens?
(A) Nothing, no information was added.
(B) Information was added, maybe?
(C) They all leave the island.
(D) They all leave the island on day 100.

On day 100, they all leave.

Sad Islanders...

Island with 100 possibly blue-eyed and green-eyed inhabitants.

Any islander who knows they have green eyes must “leave the island” that day.

No islander knows there own eye color, but knows everyone elses.

All islanders have green eyes!

First rule of island: Don’t talk about eye color!

Visitor: “I see someone has green eyes.”

Result: What happens?
(A) Nothing, no information was added.
(B) Information was added, maybe?
(C) They all leave the island.
(D) They all leave the island on day 100.

On day 100, they all leave.

Why?

They know induction.

Thm: If there are n villagers with green eyes they leave on day n.

They know induction.

Thm: If there are n villagers with green eyes they leave on day n.

Proof:
Base: $n = 1$. Person with green eyes leaves on day 1.

They know induction.

Thm: If there are n villagers with green eyes they leave on day n.

Proof:
Base: $n = 1$. Person with green eyes leaves on day 1.

Induction hypothesis:

They know induction.

Thm: If there are n villagers with green eyes they leave on day n.

Proof:
Base: $n = 1$. Person with green eyes leaves on day 1.

Induction hypothesis:
If n people with green eyes, they would leave on day n.

They know induction.

Thm: If there are n villagers with green eyes they leave on day n.

Proof:
Base: $n = 1$. Person with green eyes leaves on day 1.

Induction hypothesis:
If n people with green eyes, they would leave on day n.

Induction step:
On day $n+1$, a green eyed person sees n people with green eyes.

They know induction.

Thm: If there are n villagers with green eyes they leave on day n.

Proof:
Base: $n = 1$. Person with green eyes leaves on day 1.

Induction hypothesis:
If n people with green eyes, they would leave on day n.

Induction step:
On day $n+1$, a green eyed person sees n people with green eyes.

But they didn't leave.

They know induction.

Thm: If there are n villagers with green eyes they leave on day n.

Proof:
Base: $n = 1$. Person with green eyes leaves on day 1.

Induction hypothesis:
If n people with green eyes, they would leave on day n.

Induction step:
On day $n+1$, a green eyed person sees n people with green eyes.

But they didn't leave.

So there must be $n+1$ people with green eyes.

They know induction.

Thm: If there are n villagers with green eyes they leave on day n.

Proof:
Base: $n = 1$. Person with green eyes leaves on day 1.

Induction hypothesis:
If n people with green eyes, they would leave on day n.

Induction step:
On day $n+1$, a green eyed person sees n people with green eyes.

But they didn't leave.

So there must be $n+1$ people with green eyes.

One of them, is me.

They know induction.

Thm: If there are n villagers with green eyes they leave on day n.

Proof:
Base: $n = 1$. Person with green eyes leaves on day 1.

Induction hypothesis:
If n people with green eyes, they would leave on day n.

Induction step:
On day $n+1$, a green eyed person sees n people with green eyes.

But they didn't leave.

So there must be $n+1$ people with green eyes.

One of them, is me.

I have to leave the island.

They know induction.

Thm: If there are n villagers with green eyes they leave on day n.

Proof:
Base: $n = 1$. Person with green eyes leaves on day 1.

Induction hypothesis:
If n people with green eyes, they would leave on day n.

Induction step:
On day $n+1$, a green eyed person sees n people with green eyes.

But they didn't leave.

So there must be $n+1$ people with green eyes.

One of them, is me.

I have to leave the island. I like the island.

They know induction.

Thm: If there are n villagers with green eyes they leave on day n.

Proof:
Base: $n = 1$. Person with green eyes leaves on day 1.

Induction hypothesis:
If n people with green eyes, they would leave on day n.

Induction step:
On day $n+1$, a green eyed person sees n people with green eyes.

But they didn't leave.

So there must be $n+1$ people with green eyes.

One of them, is me.

I have to leave the island. I like the island. Sad.

They know induction.

Thm: If there are n villagers with green eyes they leave on day n.

Proof:
Base: $n = 1$. Person with green eyes leaves on day 1.

Induction hypothesis:
If n people with green eyes, they would leave on day n.

Induction step:
On day $n+1$, a green eyed person sees n people with green eyes.

But they didn't leave.

So there must be $n+1$ people with green eyes.

One of them, is me.

I have to leave the island. I like the island. Sad. $\square$

They know induction.

Thm: If there are n villagers with green eyes they leave on day n.

Proof:
Base: $n = 1$. Person with green eyes leaves on day 1.

Induction hypothesis:
If n people with green eyes, they would leave on day n.

Induction step:
On day $n+1$, a green eyed person sees n people with green eyes.

But they didn't leave.

So there must be $n+1$ people with green eyes.

One of them, is me.

I have to leave the island. I like the island. Sad. $\square$

Wait! Visitor added no information.

Common Knowledge.

Using knowledge about what other people's knowledge (your eye color) is.

Common Knowledge.

Using knowledge about what other people's knowledge (your eye color) is.

On day 1, everyone knows everyone sees more than zero.

Common Knowledge.

Using knowledge about what other people's knowledge (your eye color) is.

On day 1, everyone knows everyone sees more than zero.

Common Knowledge.

Using knowledge about what other people's knowledge (your eye color) is.

On day 1, everyone knows everyone sees more than zero.

On day 2, everyone knows everyone sees more than one.

Common Knowledge.

Using knowledge about what other people's knowledge (your eye color) is.

On day 1, everyone knows everyone sees more than zero.

On day 2, everyone knows everyone sees more than one.

Common Knowledge.

Using knowledge about what other people's knowledge (your eye color) is.

On day 1, everyone knows everyone sees more than zero.

On day 2, everyone knows everyone sees more than one.

...

Common Knowledge.

Using knowledge about what other people's knowledge (your eye color) is.

On day 1, everyone knows everyone sees more than zero.

On day 2, everyone knows everyone sees more than one.

...

On day 99, everyone knows no one sees 98

Common Knowledge.

Using knowledge about what other people's knowledge (your eye color) is.

On day 1, everyone knows everyone sees more than zero.

On day 2, everyone knows everyone sees more than one.

...

On day 99, everyone knows no one sees 98
since everyone knows everyone else does not see 97...

Common Knowledge.

Using knowledge about what other people's knowledge (your eye color) is.

On day 1, everyone knows everyone sees more than zero.

On day 2, everyone knows everyone sees more than one.

...

On day 99, everyone knows no one sees 98
since everyone knows everyone else does not see 97...

On day 100,

Common Knowledge.

Using knowledge about what other people's knowledge (your eye color) is.

On day 1, everyone knows everyone sees more than zero.

On day 2, everyone knows everyone sees more than one.

...

On day 99, everyone knows no one sees 98
since everyone knows everyone else does not see 97...

On day 100, ...uh oh!

Common Knowledge.

Using knowledge about what other people's knowledge (your eye color) is.

On day 1, everyone knows everyone sees more than zero.

On day 2, everyone knows everyone sees more than one.

...

On day 99, everyone knows no one sees 98
since everyone knows everyone else does not see 97...

On day 100, ...uh oh!

Another example:

Common Knowledge.

Using knowledge about what other people's knowledge (your eye color) is.

On day 1, everyone knows everyone sees more than zero.

On day 2, everyone knows everyone sees more than one.

...

On day 99, everyone knows no one sees 98
since everyone knows everyone else does not see 97...

On day 100, ...uh oh!

Another example:
Emperor's new clothes!

Common Knowledge.

Using knowledge about what other people's knowledge (your eye color) is.

On day 1, everyone knows everyone sees more than zero.

On day 2, everyone knows everyone sees more than one.

...

On day 99, everyone knows no one sees 98
since everyone knows everyone else does not see 97...

On day 100, ...uh oh!

Another example:
Emperor's new clothes!
No one knows other people see that he has no clothes.

Common Knowledge.

Using knowledge about what other people's knowledge (your eye color) is.

On day 1, everyone knows everyone sees more than zero.

On day 2, everyone knows everyone sees more than one.

...

On day 99, everyone knows no one sees 98
since everyone knows everyone else does not see 97...

On day 100, ...uh oh!

Another example:
Emperor's new clothes!
No one knows other people see that he has no clothes.
Until kid points it out.

Summary: principle of induction.

Today: More induction.

Summary: principle of induction.

Today: More induction.

$(P(0)$

Summary: principle of induction.

Today: More induction.

$(P(0) \wedge ((\forall k \in N)(P(k) \implies P(k+1))))$

Summary: principle of induction.

Today: More induction.

$$(P(0) \wedge ((\forall k \in N)(P(k) \implies P(k+1)))) \implies (\forall n \in N)(P(n))$$

Summary: principle of induction.

Today: More induction.

$(P(0) \wedge ((\forall k \in N)(P(k) \implies P(k+1)))) \implies (\forall n \in N)(P(n))$

Statement to prove: $P(n)$ for n starting from n_0

Summary: principle of induction.

Today: More induction.

$(P(0) \wedge ((\forall k \in N)(P(k) \implies P(k+1)))) \implies (\forall n \in N)(P(n))$

Statement to prove: $P(n)$ for n starting from n_0
Base Case: Prove $P(n_0)$.

Summary: principle of induction.

Today: More induction.

$(P(0) \wedge ((\forall k \in N)(P(k) \implies P(k+1)))) \implies (\forall n \in N)(P(n))$

Statement to prove: $P(n)$ for n starting from n_0
Base Case: Prove $P(n_0)$.
Ind. Step: Prove.

Summary: principle of induction.

Today: More induction.

$$(P(0) \wedge ((\forall k \in N)(P(k) \implies P(k+1)))) \implies (\forall n \in N)(P(n))$$

Statement to prove: $P(n)$ for n starting from n_0
Base Case: Prove $P(n_0)$.
Ind. Step: Prove. For all values, $n \geq n_0$,

Summary: principle of induction.

Today: More induction.

$(P(0) \wedge ((\forall k \in N)(P(k) \implies P(k+1)))) \implies (\forall n \in N)(P(n))$

Statement to prove: $P(n)$ for n starting from n_0
Base Case: Prove $P(n_0)$.
Ind. Step: Prove. For all values, $n \geq n_0$, $P(n) \implies P(n+1)$.

Summary: principle of induction.

Today: More induction.

$(P(0) \wedge ((\forall k \in N)(P(k) \implies P(k+1)))) \implies (\forall n \in N)(P(n))$

Statement to prove: $P(n)$ for n starting from n_0
Base Case: Prove $P(n_0)$.
Ind. Step: Prove. For all values, $n \geq n_0$, $P(n) \implies P(n+1)$.
Statement is proven!

Summary: principle of induction.

Today: More induction.

$(P(0) \wedge ((\forall k \in N)(P(k) \implies P(k+1)))) \implies (\forall n \in N)(P(n))$

Statement to prove: $P(n)$ for n starting from n_0
Base Case: Prove $P(n_0)$.
Ind. Step: Prove. For all values, $n \geq n_0$, $P(n) \implies P(n+1)$.
Statement is proven!

Strong Induction:

Summary: principle of induction.

Today: More induction.

$(P(0) \wedge ((\forall k \in N)(P(k) \implies P(k+1)))) \implies (\forall n \in N)(P(n))$

Statement to prove: $P(n)$ for n starting from n_0
Base Case: Prove $P(n_0)$.
Ind. Step: Prove. For all values, $n \geq n_0$, $P(n) \implies P(n+1)$.
Statement is proven!

Strong Induction:
$(P(0) \wedge ((\forall n \in N)(P(n) \implies P(n+1)))) \implies (\forall n \in N)(P(n))$

Summary: principle of induction.

Today: More induction.

$(P(0) \wedge ((\forall k \in N)(P(k) \implies P(k+1)))) \implies (\forall n \in N)(P(n))$

Statement to prove: $P(n)$ for n starting from n_0
Base Case: Prove $P(n_0)$.
Ind. Step: Prove. For all values, $n \geq n_0$, $P(n) \implies P(n+1)$.
Statement is proven!

Strong Induction:
$(P(0) \wedge ((\forall n \in N)(P(n) \implies P(n+1)))) \implies (\forall n \in N)(P(n))$

Also Today: strengthened induction hypothesis.

Summary: principle of induction.

Today: More induction.

$(P(0) \wedge ((\forall k \in N)(P(k) \implies P(k+1)))) \implies (\forall n \in N)(P(n))$

Statement to prove: $P(n)$ for n starting from n_0
Base Case: Prove $P(n_0)$.
Ind. Step: Prove. For all values, $n \geq n_0$, $P(n) \implies P(n+1)$.
Statement is proven!

Strong Induction:
$(P(0) \wedge ((\forall n \in N)(P(n) \implies P(n+1)))) \implies (\forall n \in N)(P(n))$

Also Today: strengthened induction hypothesis.

Strengthen theorem statement.

Summary: principle of induction.

Today: More induction.

$(P(0) \wedge ((\forall k \in N)(P(k) \implies P(k+1)))) \implies (\forall n \in N)(P(n))$

Statement to prove: $P(n)$ for n starting from n_0
Base Case: Prove $P(n_0)$.
Ind. Step: Prove. For all values, $n \geq n_0$, $P(n) \implies P(n+1)$.
Statement is proven!

Strong Induction:
$(P(0) \wedge ((\forall n \in N)(P(n) \implies P(n+1)))) \implies (\forall n \in N)(P(n))$

Also Today: strengthened induction hypothesis.

Strengthen theorem statement.
Sum of first n odds is n^2.

Summary: principle of induction.

Today: More induction.

$(P(0) \wedge ((\forall k \in N)(P(k) \implies P(k+1)))) \implies (\forall n \in N)(P(n))$

Statement to prove: $P(n)$ for n starting from n_0
Base Case: Prove $P(n_0)$.
Ind. Step: Prove. For all values, $n \geq n_0$, $P(n) \implies P(n+1)$.
Statement is proven!

Strong Induction:
$(P(0) \wedge ((\forall n \in N)(P(n) \implies P(n+1)))) \implies (\forall n \in N)(P(n))$

Also Today: strengthened induction hypothesis.

Strengthen theorem statement.
Sum of first n odds is n^2.
Hole anywhere.

Summary: principle of induction.

Today: More induction.

$(P(0) \wedge ((\forall k \in N)(P(k) \implies P(k+1)))) \implies (\forall n \in N)(P(n))$

Statement to prove: $P(n)$ for n starting from n_0
Base Case: Prove $P(n_0)$.
Ind. Step: Prove. For all values, $n \geq n_0$, $P(n) \implies P(n+1)$.
Statement is proven!

Strong Induction:
$(P(0) \wedge ((\forall n \in N)(P(n) \implies P(n+1)))) \implies (\forall n \in N)(P(n))$

Also Today: strengthened induction hypothesis.

Strengthen theorem statement.
Sum of first n odds is n^2.
Hole anywhere.
Not same as strong induction.

Summary: principle of induction.

Today: More induction.

$(P(0) \wedge ((\forall k \in N)(P(k) \implies P(k+1)))) \implies (\forall n \in N)(P(n))$

Statement to prove: $P(n)$ for n starting from n_0
Base Case: Prove $P(n_0)$.
Ind. Step: Prove. For all values, $n \geq n_0$, $P(n) \implies P(n+1)$.
Statement is proven!

Strong Induction:
$(P(0) \wedge ((\forall n \in N)(P(n) \implies P(n+1)))) \implies (\forall n \in N)(P(n))$

Also Today: strengthened induction hypothesis.

Strengthen theorem statement.
Sum of first n odds is n^2.
Hole anywhere.
Not same as strong induction. E.g., used in product of primes proof.

Summary: principle of induction.

Today: More induction.

$(P(0) \wedge ((\forall k \in N)(P(k) \implies P(k+1)))) \implies (\forall n \in N)(P(n))$

Statement to prove: $P(n)$ for n starting from n_0
Base Case: Prove $P(n_0)$.
Ind. Step: Prove. For all values, $n \geq n_0$, $P(n) \implies P(n+1)$.
Statement is proven!

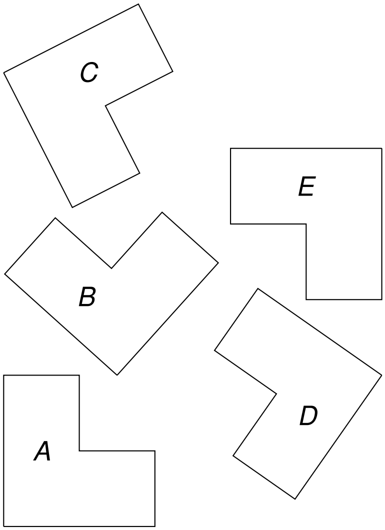
Strong Induction:
$(P(0) \wedge ((\forall n \in N)(P(n) \implies P(n+1)))) \implies (\forall n \in N)(P(n))$

Also Today: strengthened induction hypothesis.

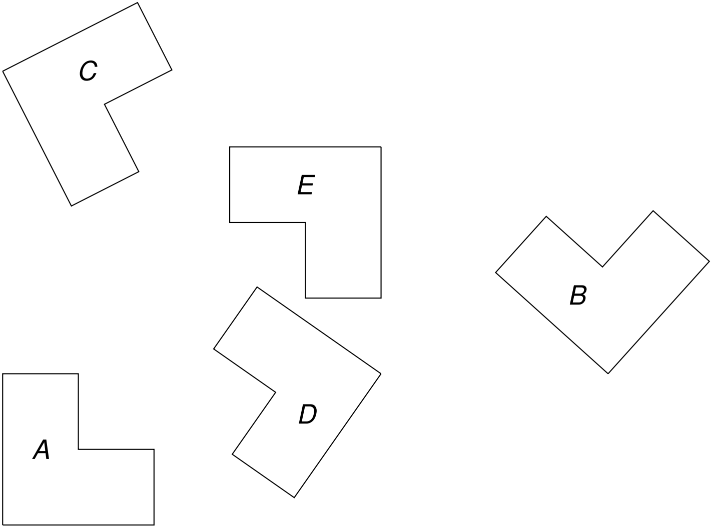
Strengthen theorem statement.
Sum of first n odds is n^2.
Hole anywhere.
Not same as strong induction. E.g., used in product of primes proof.

Induction $\equiv$ Recursion.

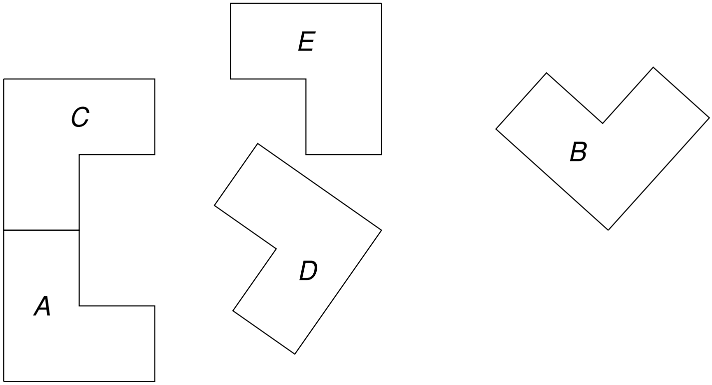
Tiling Cory Hall Courtyard.



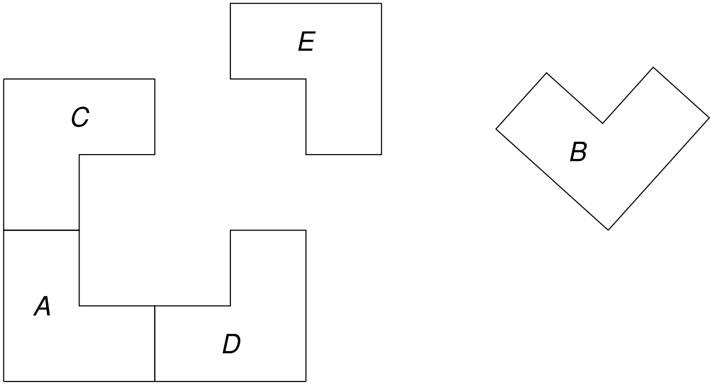
Tiling Cory Hall Courtyard.



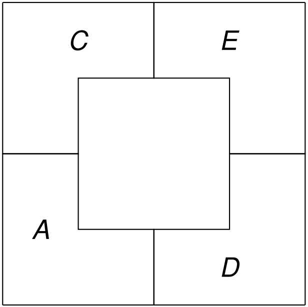
Tiling Cory Hall Courtyard.

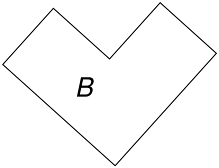


Tiling Cory Hall Courtyard.

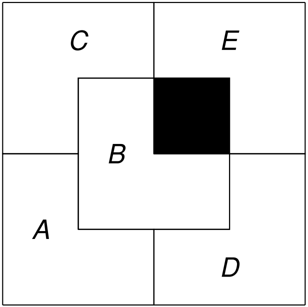


Tiling Cory Hall Courtyard.





Tiling Cory Hall Courtyard.



Tiling Cory Hall Courtyard.