Climb an infinite ladder?

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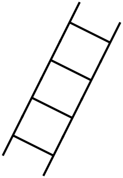
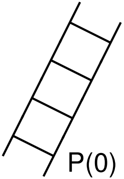


Climb an infinite ladder?

$P(0)$



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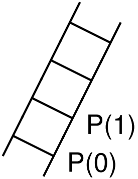
$$\forall k, P(k) \implies P(k+1)$$



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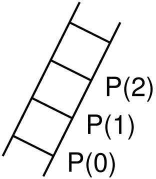
$$P(0) \implies P(1) \implies P(2)$$



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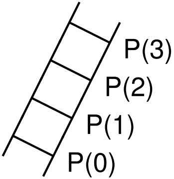

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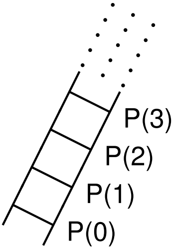

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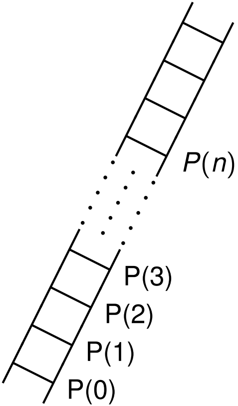

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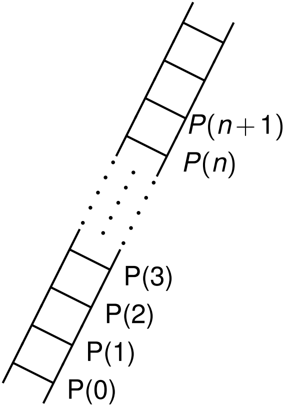

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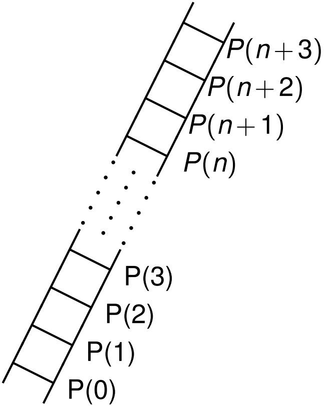

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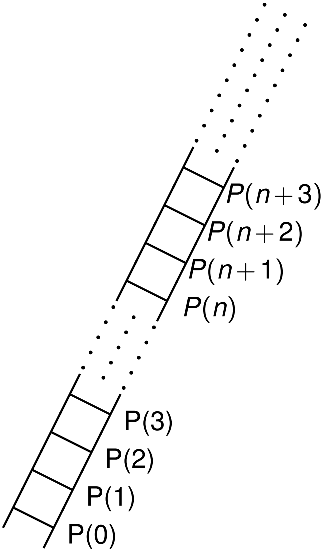

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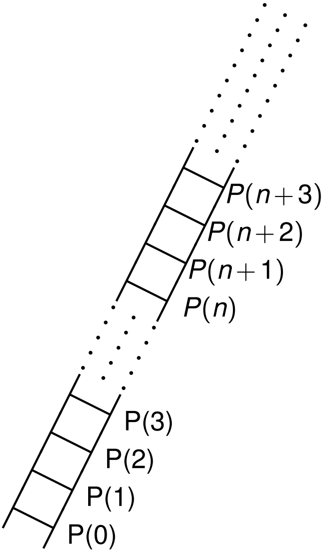

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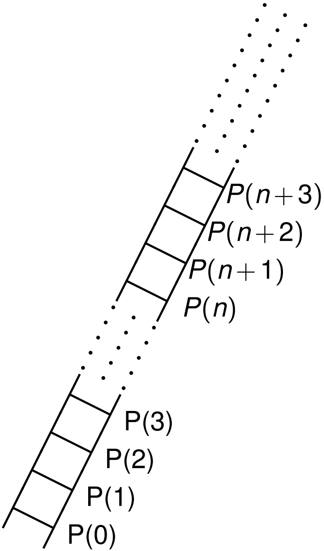

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Island with 100 possibly blue-eyed and green-eyed inhabitants.

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Thm: If there are n villagers with green eyes they leave on day n.

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Wait! Visitor added no information.

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If 66 villagers out of the 100 had green eyes, what would happen?

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Strong Induction and Recursion.

Thm: For every natural number $n \geq 12$, $n = 4x + 5y$.

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     (x',y') = find-x-y(n-4)
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Prove: Given n, returns (x,y) where $n = 4x + 5y$, for $n \geq 12$.

Base cases: P(12) , P(13) , P(14) , P(15). Yes.

Strong Induction and Recursion.

Thm: For every natural number $n \geq 12$, $n = 4x + 5y$.

Instead of proof, let's write some code!

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Slight differences: showed for all $n \geq 16$ that $\wedge_{i=4}^{n-1} P(i) \implies P(n)$.

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The induction principle works on the natural numbers.

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Proves statements of form: $\forall n \in \mathbb{N}, P(n)$.

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Any set where any subset of the set has a smallest element.

In some sense, the natural numbers.

Stable Matching Problem

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- n candidates and n jobs.

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- *n* candidates and *n* jobs.
- Each job has a ranked preference list of candidates.

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- Maximize number of first choices.
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- Minimize difference between preference ranks.

The best laid plans..

Consider the pairs..

- Cal Bears and the Pac-12
- Wake Forest and the ACC

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Cal Bears prefers the ACC

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The ACC prefers Cal Bears.

Uh..oh.

The best laid plans..

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The ACC prefers Cal Bears.

Uh..oh. Sad Pac-12, (and Wake Forest.)

So..

Produce a matching where there are no crazy moves!

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Definition: A **matching** is disjoint set of n job-candidate pairs.

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Definition: A **matching** is disjoint set of *n* job-candidate pairs.

Example: A matching
$S = \{(CalBears, Pac-12); (WakeForest, ACC)\}$.

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b and g^* prefer each other to their partners in S

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Definition: A **rogue couple** b, g^* for a pairing S:
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Example: Cal Bears and the ACC are a rogue couple in S.

Not a great example of stable matching, but interesting exercise in "selfish" incentives.

A stable matching??

Given a set of preferences.

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Is there a stable matching?

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How does one find it?

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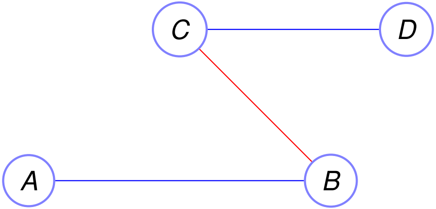
Consider a single type version: stable roommates.

A	B	C	D
B	C	A	D
C	A	B	D
D	A	B	C



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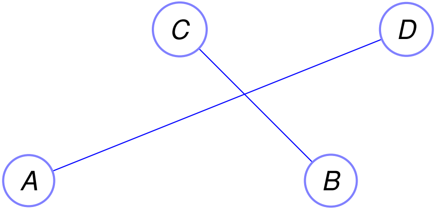

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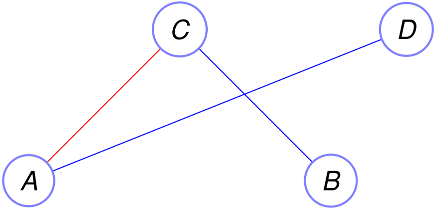

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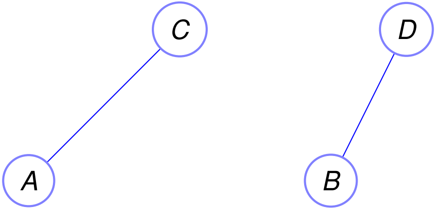

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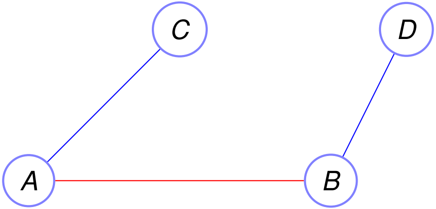

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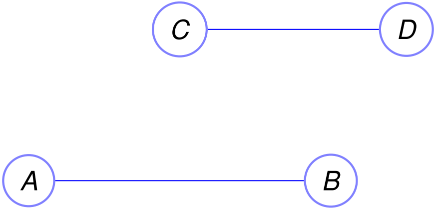

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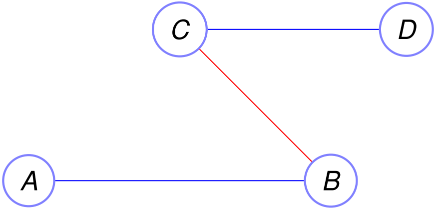

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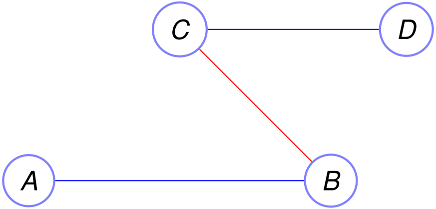

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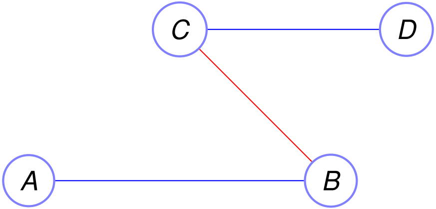

The Propose and Reject Algorithm.

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Each Day:

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Stop when each job gets exactly one proposal.

Example.

Jobs			
A	1	2	3
B	1	2	3
C	2	1	3

Candidates			
1	C	A	B
2	A	B	C
3	A	C	B

Example.

Jobs			
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B	1	2	3
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Candidates			
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	Day 1	Day 2	Day 3	Day 4	Day 5
1					
2					
3					

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Jobs			
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Candidates			
1	C	A	B
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3	A	C	B

	Day 1	Day 2	Day 3	Day 4	Day 5
1	A, B				
2	C				
3					

Example.

	Jobs		
A	1	2	3
B	~~1~~	2	3
C	2	1	3

	Candidates		
1	C	A	B
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	Day 1	Day 2	Day 3	Day 4	Day 5
1	A, ~~B~~				
2	C				
3					

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	Day 1	Day 2	Day 3	Day 4	Day 5
1	A, ~~B~~	A			
2	C	B, C			
3					

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3					

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	Day 1	Day 2	Day 3	Day 4	Day 5
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3					B

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2	C	B, C	B	A,B	A
3					B

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Do jobs or candidates do “better”?

The Propose and Reject Algorithm.

Each Day:

1. Each job **proposes** to its favorite candidate on its list.
2. Each candidate rejects all but their favorite proposer
   (whom they put on a string.)
3. Rejected job crosses rejecting candidate off its list.

Stop when each job gets exactly one proposal.
Does this terminate?

...produce a matching?

....a stable matching?

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Termination.

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Total size of lists?

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Total size of lists? *n* jobs, *n* length list.

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Terminates in $\leq n^2$ steps!

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(A) By induction on days.

(B) When the economy is good.

(C) The candidate can always keep the job on the string.

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Economics: Study of choice.

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Poll: The argument for termination ...

(A) Implies: no unmatched job at end.

(B) Uses Improvement Lemma: every candidate matched.

(C) From Algorithm: unmatched job would ask everyone.

(D) Implies: every one gets their favorite job.

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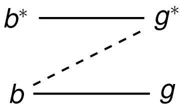

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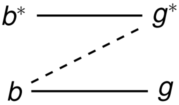

b prefers g^* to g.

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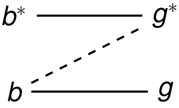

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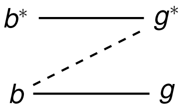

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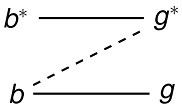

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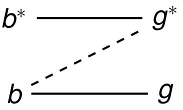

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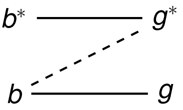

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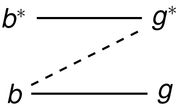

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(A) Contradiction.

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Good for jobs? candidates?

Is the Job-Proposes better for jobs?

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A:	1,2	1:	A,B
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Which is optimal for A?

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Which is optimal for A? S

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Which is optimal for 1?

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A:	1,2	1:	A,B
B:	1,2	2:	B,A

Consider pairing: $(A,1),(B,2)$.

Stable? Yes.

Optimal for *B*?
Notice: only one stable pairing. If (A,2) are pair, (A,1) is rogue couple.
So this is the best *B* can do in a stable pairing.
So optimal for *B*.

Also optimal for *A*, 1 and 2. Also pessimal for *A*,*B*,1 and 2.

A:	1,2	1:	B,A
B:	2,1	2:	A,B

Pairing *S*: $(A,1),(B,2)$. Stable? Yes.

Pairing *T*: $(A,2),(B,1)$. Also Stable.

Which is optimal for *A*? *S* Which is optimal for *B*? *S*
Which is optimal for 1? *T* Which is optimal for 2? *T*

Pessimality?

Job Propose and Candidate Reject is optimal!

For jobs?

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For jobs? For candidates?

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Theorem: Job Propose and Reject produces a job-optimal pairing.

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Proof:

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Structural statement: Job optimality $\Longrightarrow$ Candidate pessimality.

Quick Questions.

How does one make it better for candidates?

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Propose and Reject - stable matching algorithm. One side proposes.

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Residency Matching..

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The method was used to match residents to hospitals.

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Hospital optimal....

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Another variation: couples.

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Improvement Lemma plus every day the job gets to choose.
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Job Optimality:
contradiction of the existence of a better *stable* pairing.
that is, no rogue couple by improvement, job choice, and well ordering principle.
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