Today.

Quick review.

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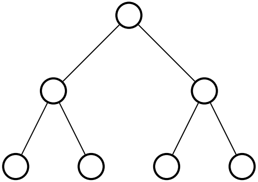
Finish Graphs (mostly)

A Tree, a tree.

Graph $G = (V, E)$.

Binary Tree!



More generally.

Trees.

Definitions:

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A connected graph without a cycle.

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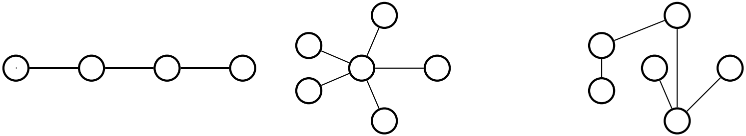

no cycle and connected?

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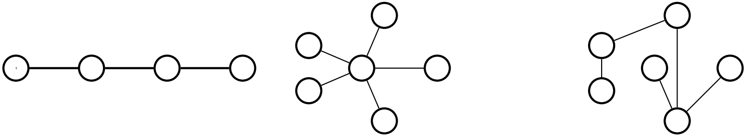

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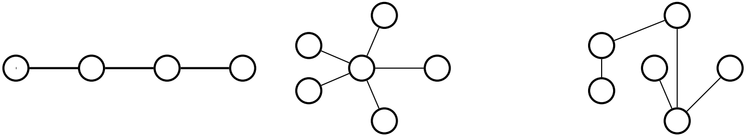

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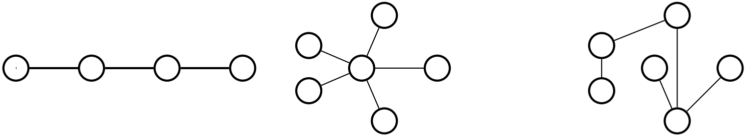

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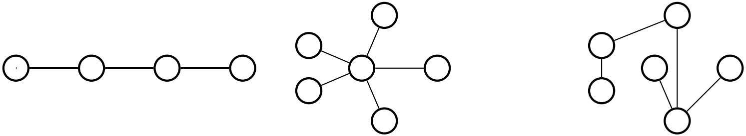

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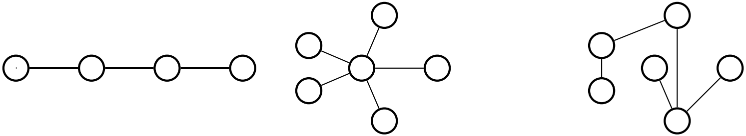

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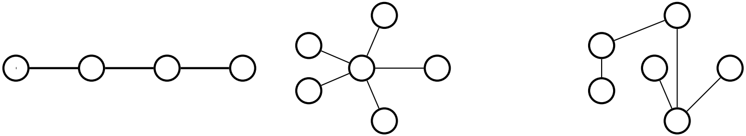

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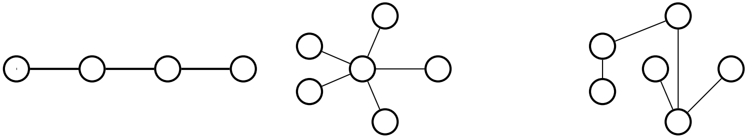

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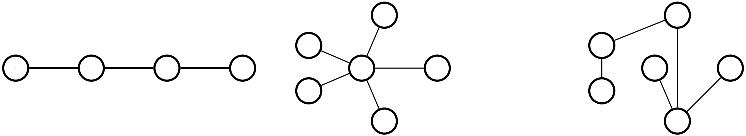

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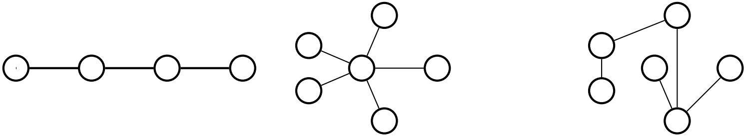

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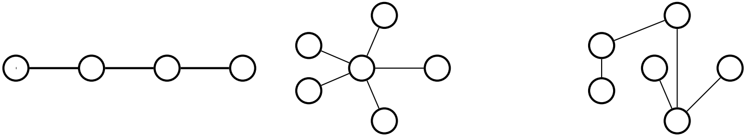

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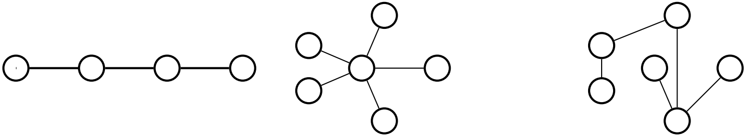

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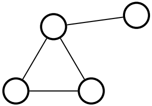
To tree or not to tree!



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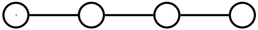
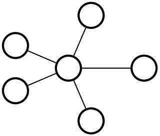
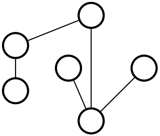

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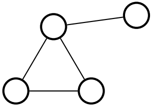
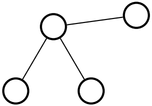

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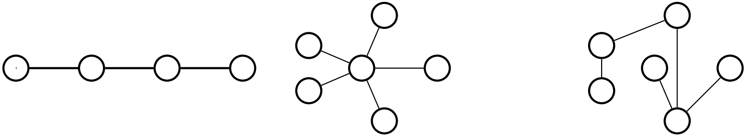

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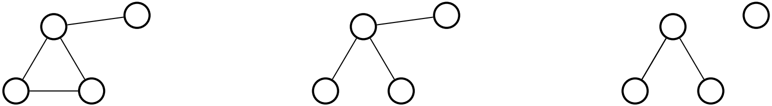

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Theorem:
"G connected and has $|V| - 1$ edges" $\equiv$
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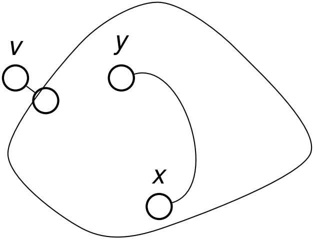

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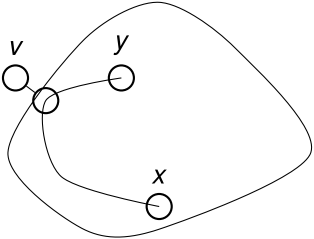

Proof of only if.

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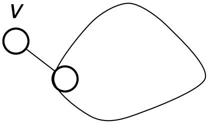

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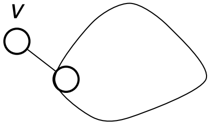

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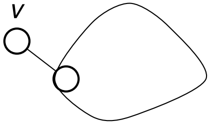

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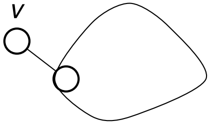

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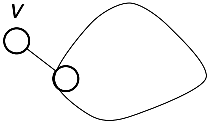

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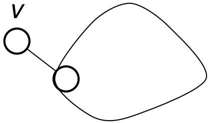

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Claim: There is a degree 1 node.

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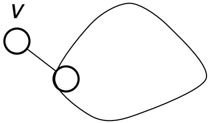

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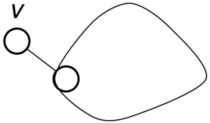

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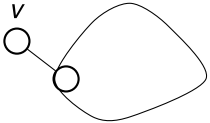

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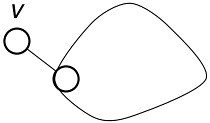

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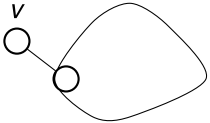

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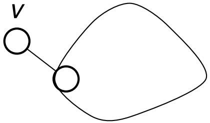

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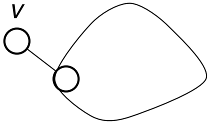

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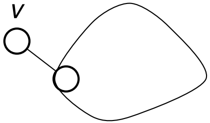

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$G-v$ has $|V|-1$ vertices and $|V|-2$ edges so by induction

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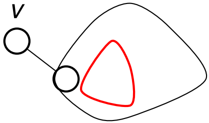

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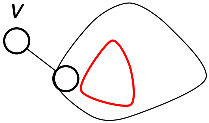
"G connected and has $|V| - 1$ edges" $\implies$
"G is connected and has no cycles."

Proof of $\implies$: By induction on $|V|$.
Base Case: $|V| = 1$. $0 = |V| - 1$ edges and has no cycles.

Induction Step:
Claim: There is a degree 1 node.
Proof: First, connected $\implies$ every vertex degree ≥ 1.
Sum of degrees is $2|E| = 2(|V| - 1) = 2|V| - 2$
Average degree $(2|V| - 2)/|V| = 2 - (2/|V|)$. Must be a degree 1 vertex.
Cuz not everyone is bigger than average! □
By degree 1 removal lemma, $G - v$ is connected.
$G - v$ has $|V| - 1$ vertices and $|V| - 2$ edges so by induction
$\implies$ no cycle in $G - v$.

Proof of only if.



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And no cycle in G since degree 1 cannot participate in cycle.

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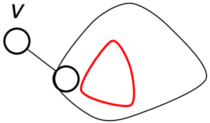

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Walk from a vertex using untraversed edges.

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Can't visit more than once since no cycle.

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New graph is connected.

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Poll: Oh tree, beautiful tree.

Let G be a connected graph with $|V|-1$ edges.

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Lecture Summary.

Graphs.

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Basics.

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Degree, Incidence, Sum of degrees is $2|E|$. Connectivity.

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Graphs.
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    - maximal set of vertices that are connected.

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(Ind. Hyp.)
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Proof of “handshake” lemma.

Lemma: The sum of degrees is $2|E|$, for a graph $G = (V, E)$.
What’s true?

(A) The number of edge-vertex incidences for an edge e is 2.
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Handshake lemma: sum of number of handshakes of each person is twice the number of handshakes.

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A graph is Euleurian if it is connected and has even degree.

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(F) A tour connecting a set of connected components, each with a Eulerian tour is really cool! This implies the graph is Eulerian.

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Only (D) is false. The rest are steps in the proof.

Lecture 6.

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Euler's Formula.

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Planar Six and then Five Color theorem.

Lecture 6.

Euler's Formula.

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Types of graphs.

Lecture 6.

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Complete Graphs.
Trees (a little more.)
Hypercubes.

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A graph that can be drawn in the plane without edge crossings.

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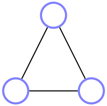

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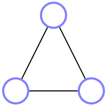

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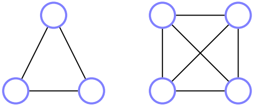

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Four node complete?

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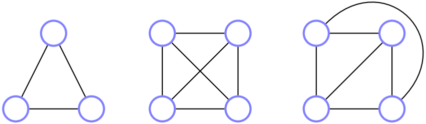

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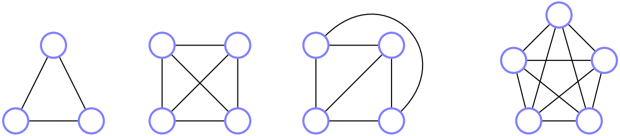

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Five node complete or K_5 ?

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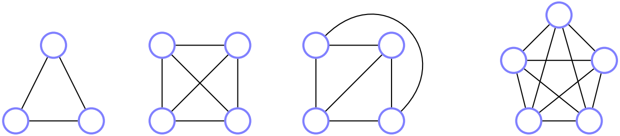

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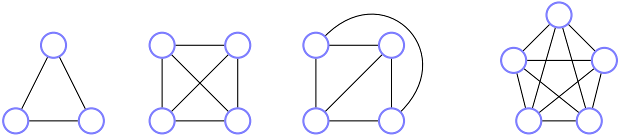

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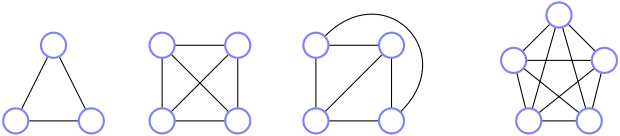

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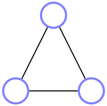 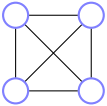 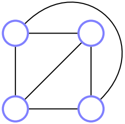 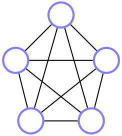

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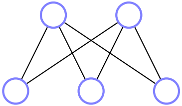

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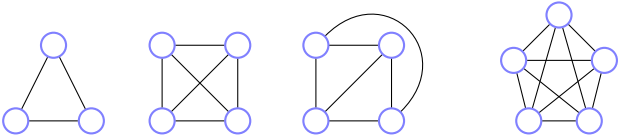

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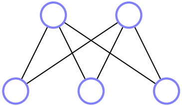

Two to three nodes, bipartite?

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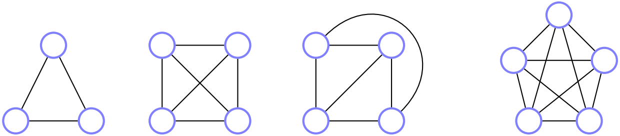

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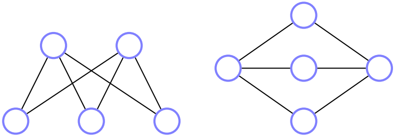

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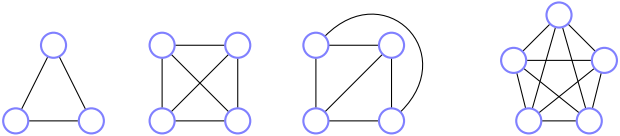

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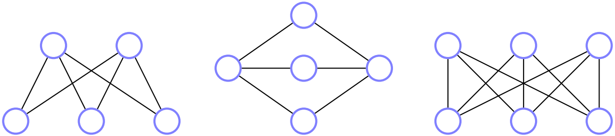

Two to three nodes, bipartite? Yes.
Three to three nodes, complete/bipartite or $K_{3,3}$.

Planar graphs.

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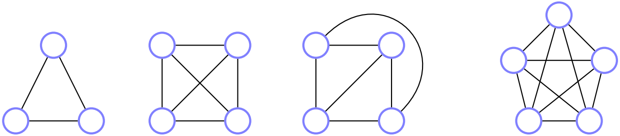
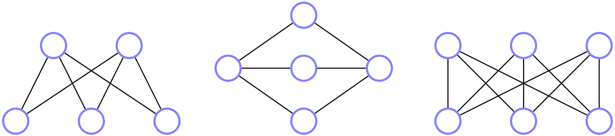


Planar? Yes for Triangle.
Four node complete? Yes.
(complete $\equiv$ every edge present. K_n is n-vertex complete graph.)
Five node complete or K_5 ? No! Why? Later.



Two to three nodes, bipartite? Yes.
Three to three nodes, complete/bipartite or $K_{3,3}$. No.

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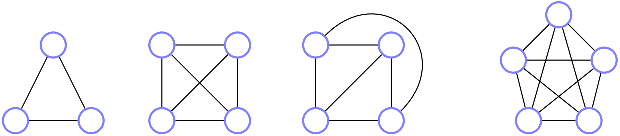

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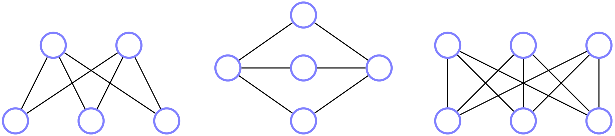

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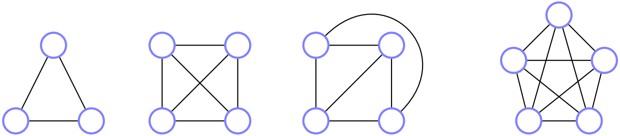

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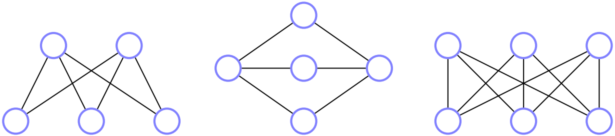

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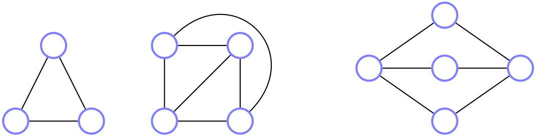

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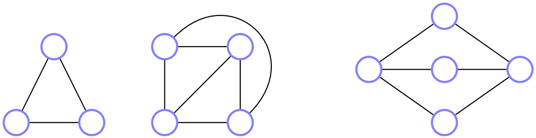
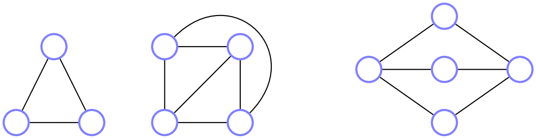


Faces: connected regions of the plane.

Euler's Formula.



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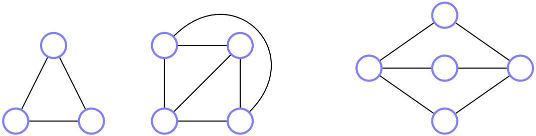

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How many faces for
triangle?

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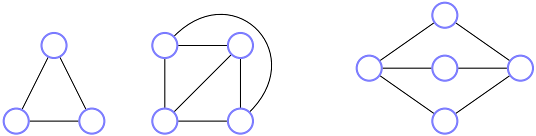

Faces: connected regions of the plane.

How many faces for
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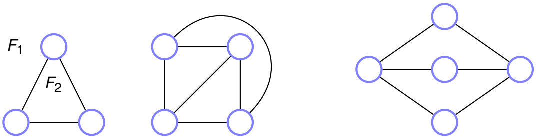

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How many faces for

triangle? 2

complete on four vertices or K_4?

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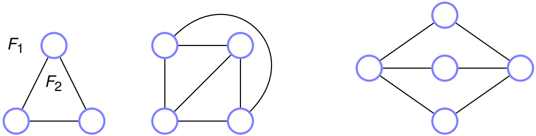

Faces: connected regions of the plane.

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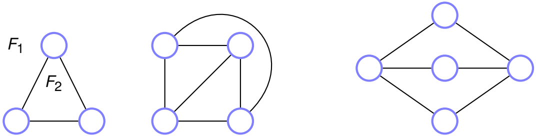

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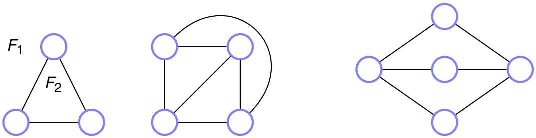

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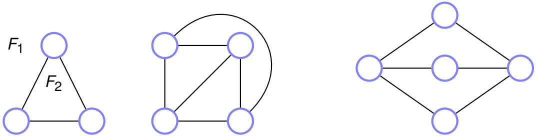

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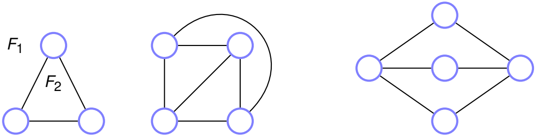

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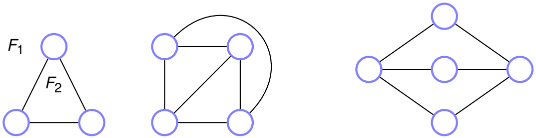

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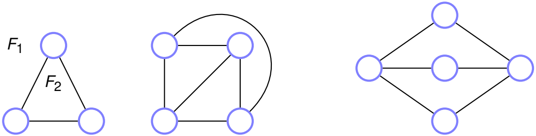

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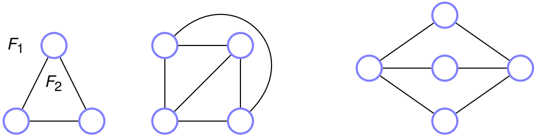

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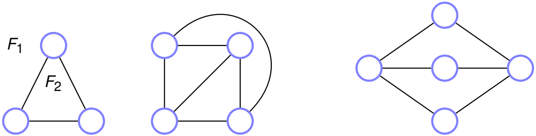

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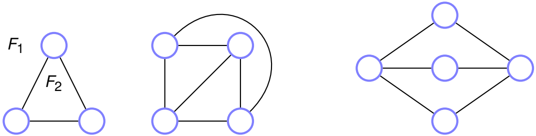

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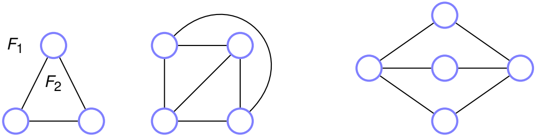

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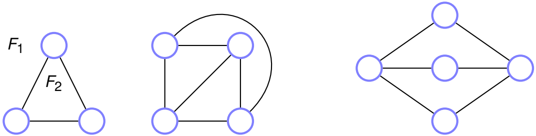

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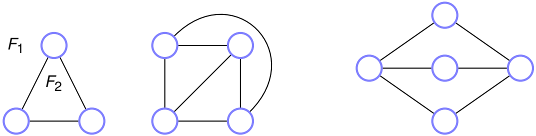

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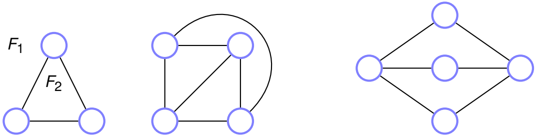

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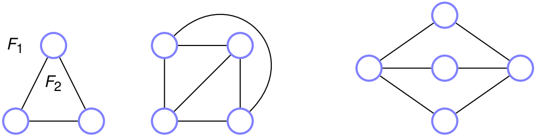

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Examples = 3!

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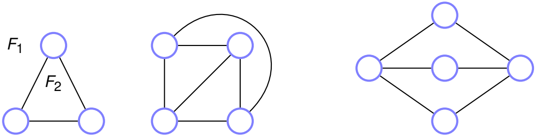

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Examples = 3! Proven!

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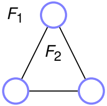

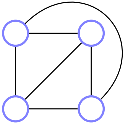

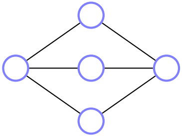

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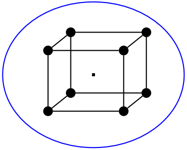

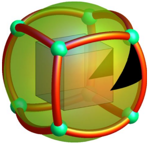

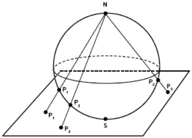

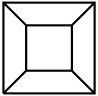

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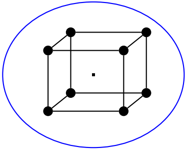
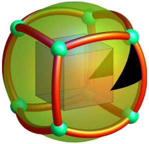
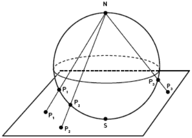
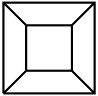
Greeks knew formula for polyhedron.









Faces?

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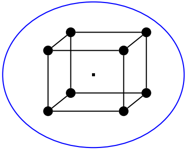

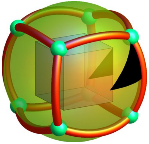

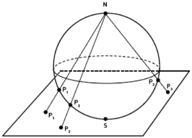

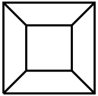

Faces? 6.   Edges?

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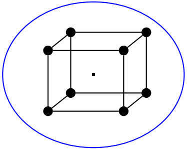

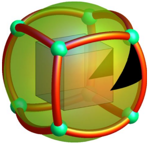

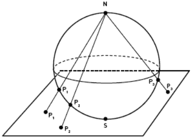

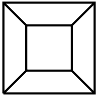

Faces? 6. Edges? 12.

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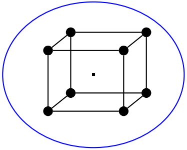
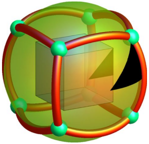
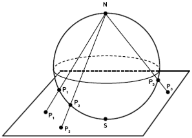

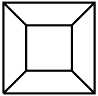

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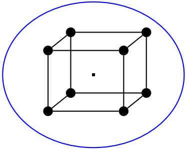

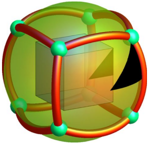

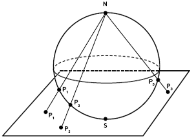

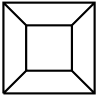

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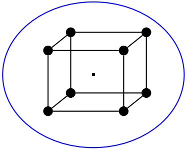
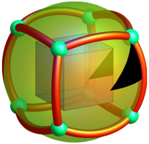
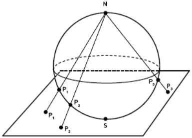
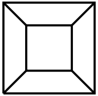
Euler and Polyhedron.

Greeks knew formula for polyhedron.



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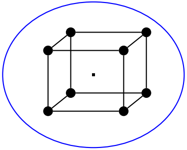
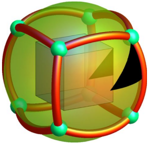
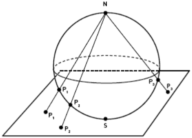

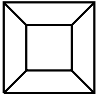

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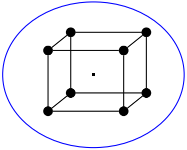
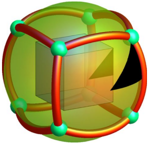
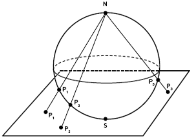

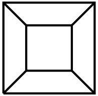

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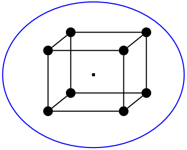

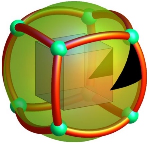

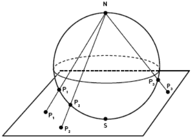

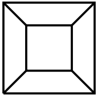

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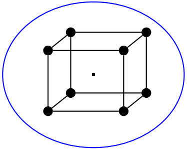
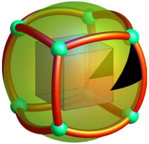
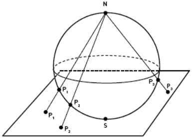

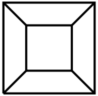

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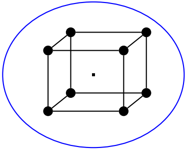
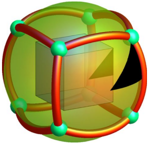
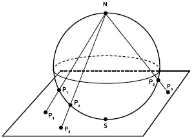

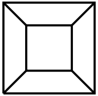

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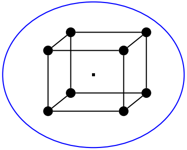

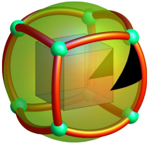

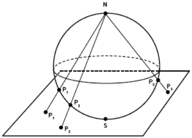

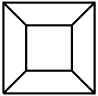

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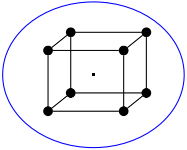
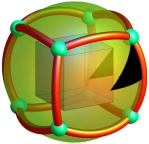
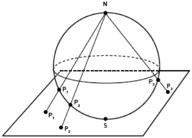
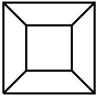
Polyhedron without holes

Euler and Polyhedron.

Greeks knew formula for polyhedron.



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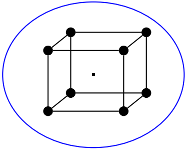
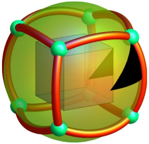
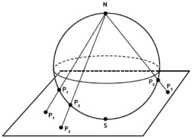

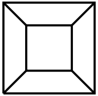

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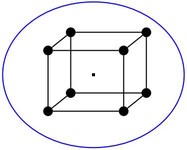
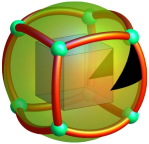
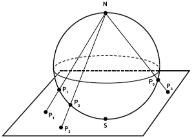

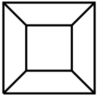

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For Convex Polyhedron:

Euler and Polyhedron.

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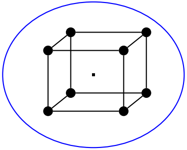
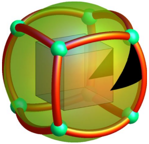
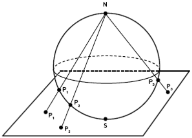

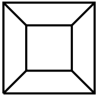

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For Convex Polyhedron:

Surround by sphere.

Euler and Polyhedron.

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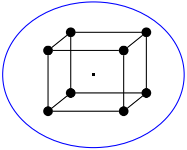
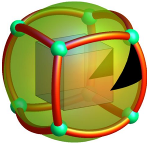
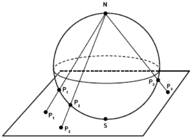

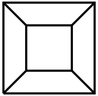

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For Convex Polyhedron:

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Project from internal point polytope to sphere:

Euler and Polyhedron.

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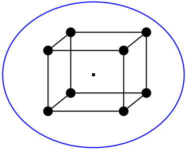
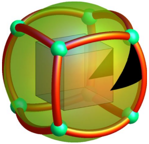
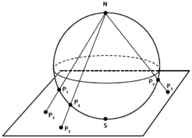

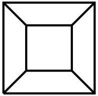

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For Convex Polyhedron:

Surround by sphere.

Project from internal point polytope to sphere: drawing on sphere.

Euler and Polyhedron.

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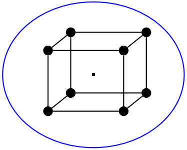

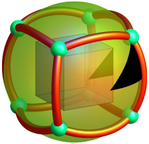

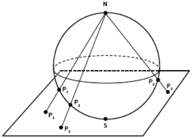

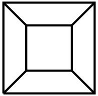

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For Convex Polyhedron:

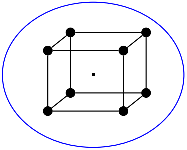
Surround by sphere.

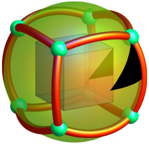
Project from internal point polytope to sphere: drawing on sphere.

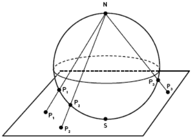
Project Sphere-N

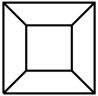
Euler and Polyhedron.

Greeks knew formula for polyhedron.









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For Convex Polyhedron:

Surround by sphere.

Project from internal point polytope to sphere: drawing on sphere.

Project Sphere-N onto Plane:

Euler and Polyhedron.

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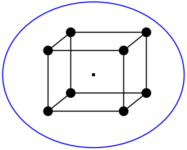

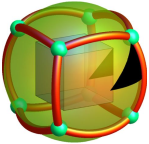

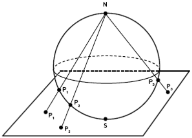

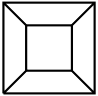

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Euler: Connected planar graph: $v + f = e + 2$.

$8 + 6 = 12 + 2$.

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Polyhedron without holes $\equiv$ Planar graphs.

For Convex Polyhedron:

Surround by sphere.

Project from internal point polytope to sphere: drawing on sphere.

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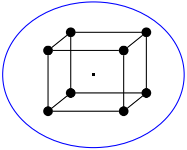

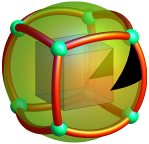

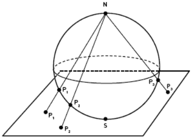

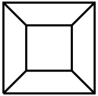

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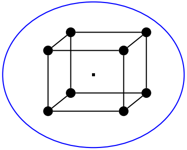

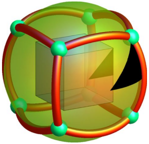

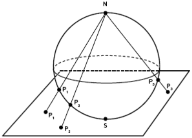

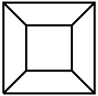

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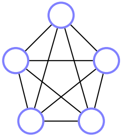
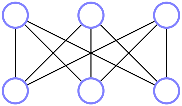
Euler proved formula thousands of years later!

Euler and non-planarity of K_5 and $K_{3,3}$





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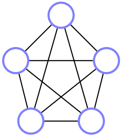

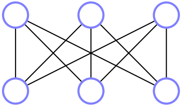

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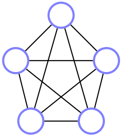
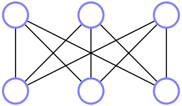

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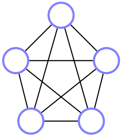

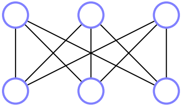

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Consider Face edge Adjacencies with multiplicities

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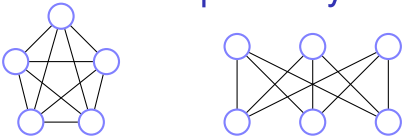

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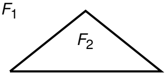

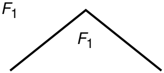

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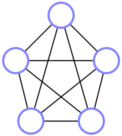

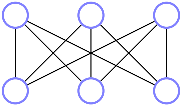

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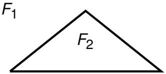

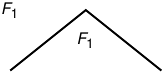

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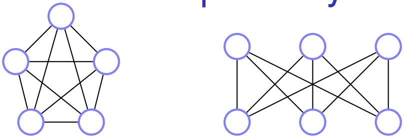

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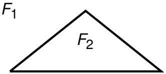

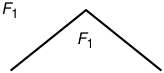

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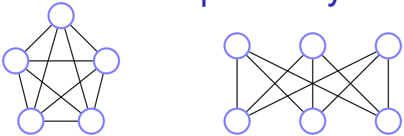

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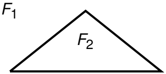

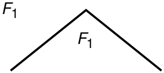

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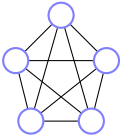

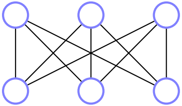

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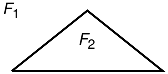

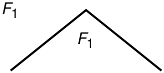

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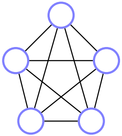

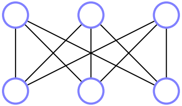

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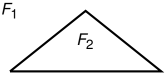

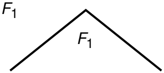

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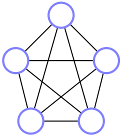

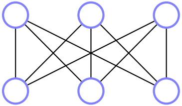

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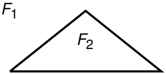

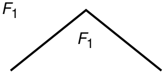

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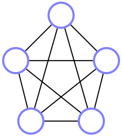

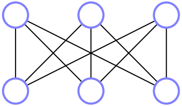

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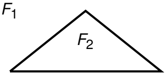

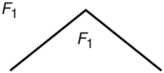

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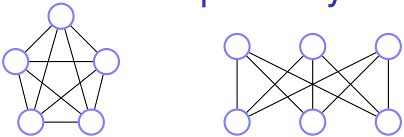

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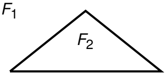

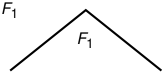

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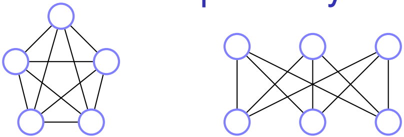

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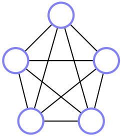

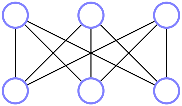

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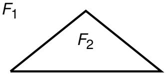

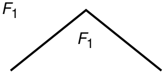

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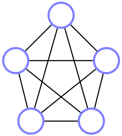

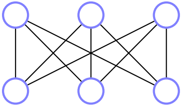

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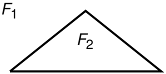

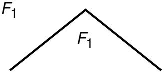

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$\geq 3f$ face-edge adjacencies.

Each edge is adjacent to two faces.

$= 2e$ face-edge adjacencies.

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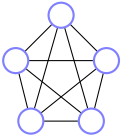
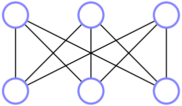
Plug into Euler:

Euler and non-planarity of K_5 and $K_{3,3}$





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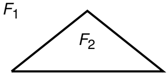

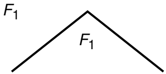

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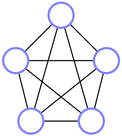
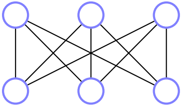
Euler and non-planarity of K_5 and $K_{3,3}$





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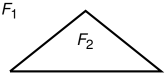

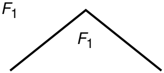

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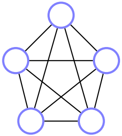

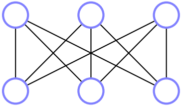

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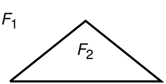

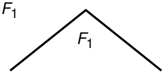

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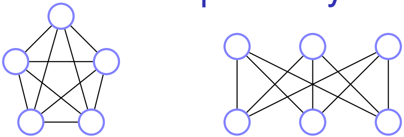
K_5

Euler and non-planarity of K_5 and $K_{3,3}$



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Consider Face edge Adjacencies with multiplicities

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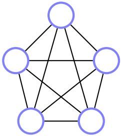
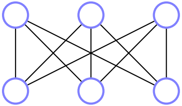
K_5 Edges?

Euler and non-planarity of K_5 and $K_{3,3}$





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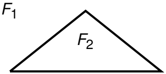

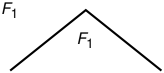

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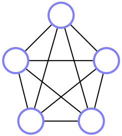
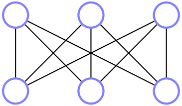
Euler and non-planarity of K_5 and $K_{3,3}$





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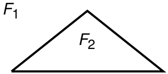

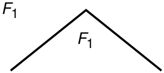

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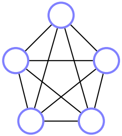
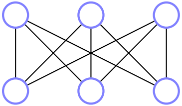
Euler and non-planarity of K_5 and $K_{3,3}$





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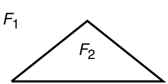

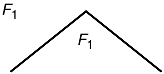

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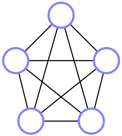
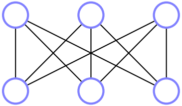
Euler and non-planarity of K_5 and $K_{3,3}$





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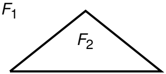

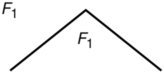

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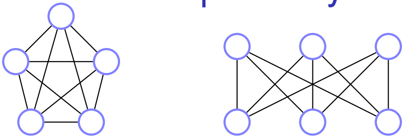
Euler and non-planarity of K_5 and $K_{3,3}$



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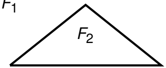

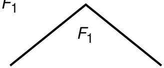

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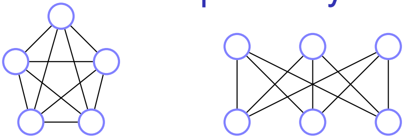
Euler and non-planarity of K_5 and $K_{3,3}$



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$10 \not\leq 3(5) - 6 = 9$. $\implies K_5$ is not planar.

Planar $\implies e \leq 3v - 6$. Flow Poll.

Euler's formula: $v + f = e + 2$

Consider graph with > 2 vertices. Understand the following.

(A) Every face is incident to ≥ 3 edges.

(B) $\|$ Face-edge incidences $\| \geq 3f$

(C) Every edge is incident (with multiplicity) to 2 faces.

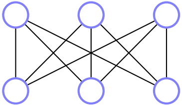
(D) $\|$Face edge incidences$\| = 2e$

(E) $3f \leq \|$Face-ege-incidences$\| = 2e$

(F) $3(e + 2 - v) <= 2e$

Conclusion: $e <= 3v - 6$

Proving non-planarity for $K_{3,3}$



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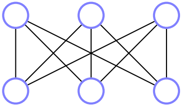
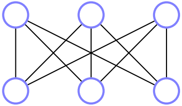


$K_{3,3}$?

Proving non-planarity for $K_{3,3}$



$K_{3,3}$? Edges?

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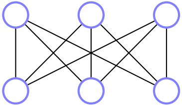

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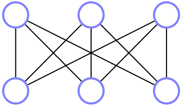

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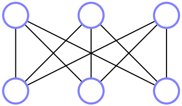

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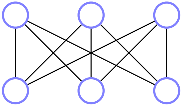
$e \leq 3(v) - 6$ for planar graphs.

Proving non-planarity for $K_{3,3}$



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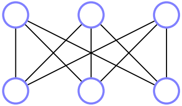

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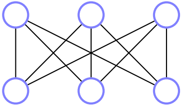

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Step in proof of K_5: faces are adjacent to ≥ 3 edges.

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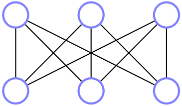

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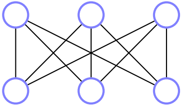
For $K_{3,3}$ every cycle is of even length or incident ≥ 4 faces.

Proving non-planarity for $K_{3,3}$



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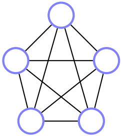
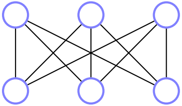
Finish in homework!

Planarity and Euler





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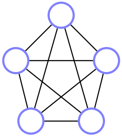
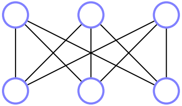
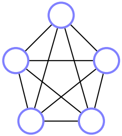
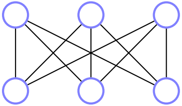




These graphs **cannot** be drawn in the plane without edge crossings.

Planarity and Euler





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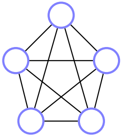

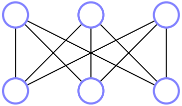

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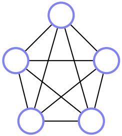

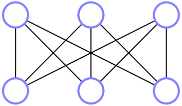

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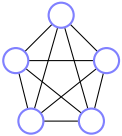

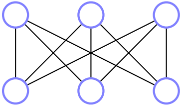

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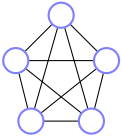

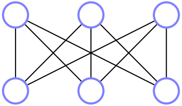

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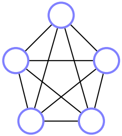

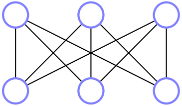

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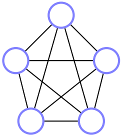

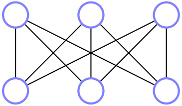

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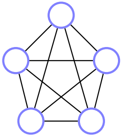

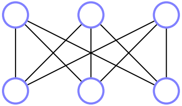

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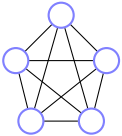

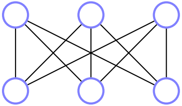

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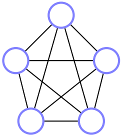

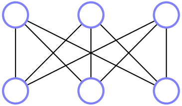

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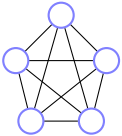

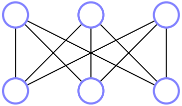

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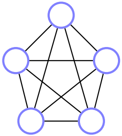
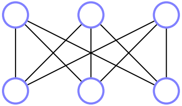
Planarity and Euler





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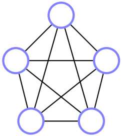

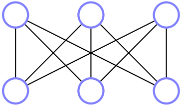

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So......so ...Cool!

Euler's formula.

Euler: Connected planar graph has $v + f = e + 2$.

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Proof:

Euler's formula.

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Proof: Induction on e.
Base:

Euler's formula.

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Proof: Induction on e.
Base: $e = 0$,

Euler's formula.

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Proof: Induction on e.
Base: $e = 0$, $v = f = 1$.

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Proof: Induction on e.
Base: $e = 0$, $v = f = 1$.
Induction Step:

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Euler: Connected planar graph has $v+f=e+2$.

Proof: Induction on e.
Base: $e=0$, $v=f=1$.
Induction Step:
If it is a tree.

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Proof: Induction on e.
Base: $e=0$, $v=f=1$.
Induction Step:
If it is a tree. $e=v-1$, $f=1$, $v+1=(v-1)+2$. Yes.

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If not a tree.
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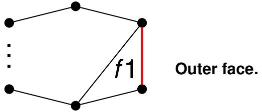

Joins two faces.

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Proof: Induction on e.
Base: $e = 0$, $v = f = 1$.
Induction Step:
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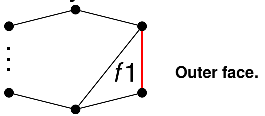

Joins two faces.
New graph: v-vertices.

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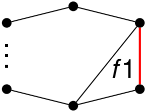

Outer face.

Joins two faces.
New graph: v-vertices. $e-1$ edges.

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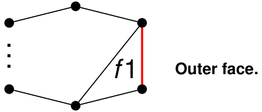

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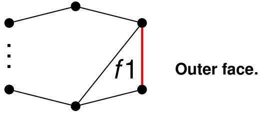

Joins two faces.
New graph: v-vertices. $e - 1$ edges. $f - 1$ faces. Planar.

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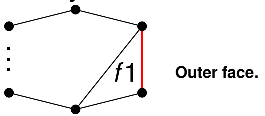

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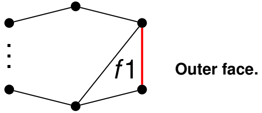

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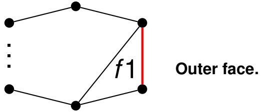

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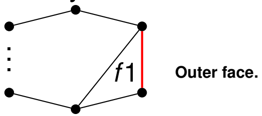

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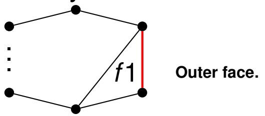

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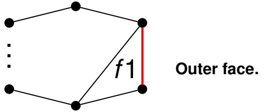

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Euler: $v+f=e+2$.
Tree satisfies formula: $v+1=(v-1)+2$

Euler's formula.

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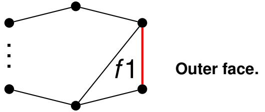

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Euler: $v+f=e+2$.
Tree satisfies formula: $v+1=(v-1)+2$
adding edge adds face: $e\to e+1$, $f\to f+1$.

Euler's Proof.Poll.

Euler: Connected planar graph has $v + f = e + 2$.

Steps/concepts in proof of euler's formula.

Euler's Proof.Poll.

Euler: Connected planar graph has $v+f=e+2$.

Steps/concepts in proof of euler's formula.

(A) Planar drawing of tree has 1 face.
(B) Tree has $|V|-1$ edges.
(C) Induction.
(D) face is adjacent to at least 3 edges.
(E) edge has two edge-vertex incidences.
(F) Add edge to planar drawing splits a face.

Euler's Proof.Poll.

Euler: Connected planar graph has $v+f=e+2$.

Steps/concepts in proof of euler's formula.

(A) Planar drawing of tree has 1 face.
(B) Tree has $|V|-1$ edges.
(C) Induction.
(D) face is adjacent to at least 3 edges.
(E) edge has two edge-vertex incidences.
(F) Add edge to planar drawing splits a face.

All are true and all are relevant to the proof, though (E) is more analagous than direct.

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Given $G = (V, E)$, a coloring of G assigns colors to vertices V where for each edge the endpoints have different colors.

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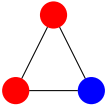

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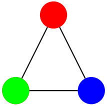

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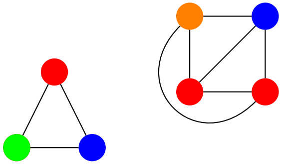

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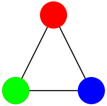

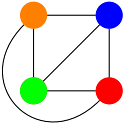

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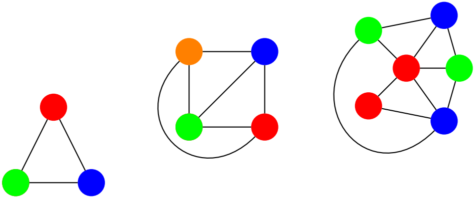

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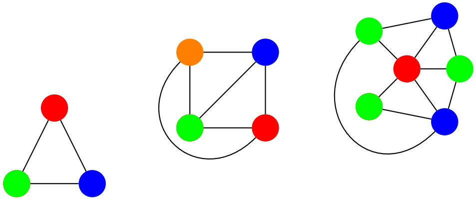

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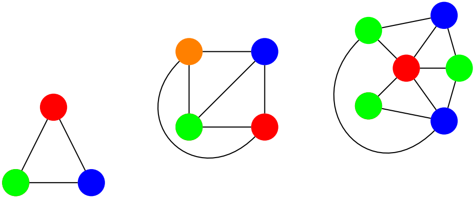

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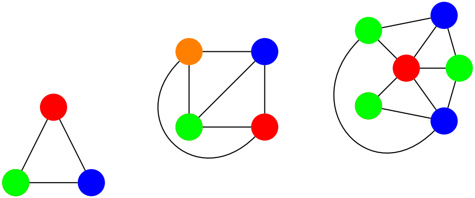

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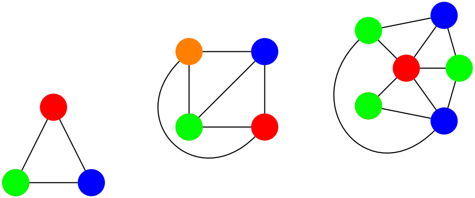

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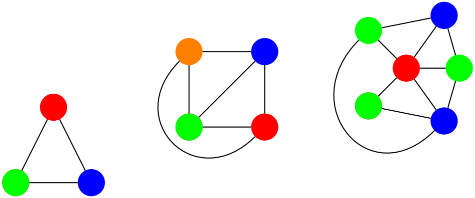

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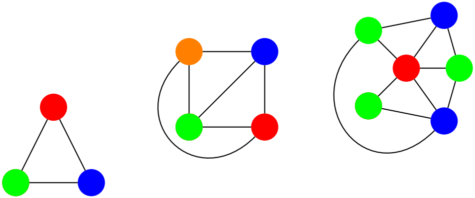

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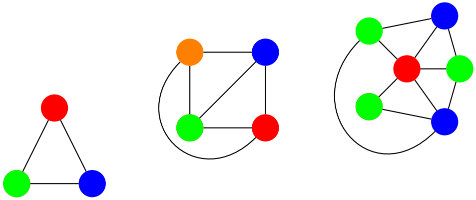

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Interesting things to do.

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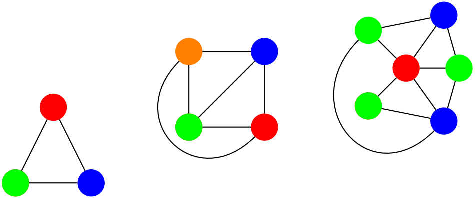

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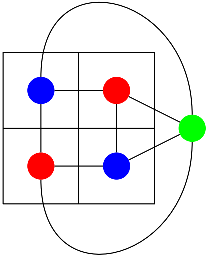
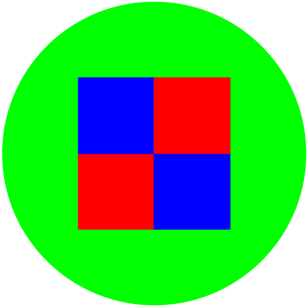
Interesting things to do. Algorithm!

Planar graphs and maps.

Planar graph coloring $\equiv$ map coloring.





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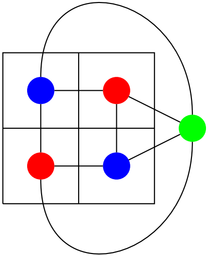

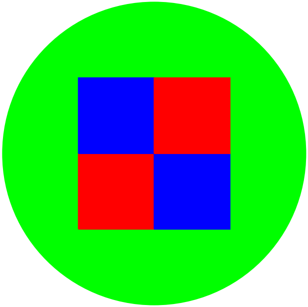

Four color theorem is about planar graphs!

Six color theorem.

Theorem: Every planar graph can be colored with six colors.

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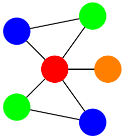
Five color theorem: prelimnary.

Preliminary Observation: Connected components of vertices with two colors in a legal coloring can switch colors.



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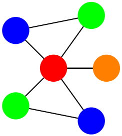

Look at only green and blue.

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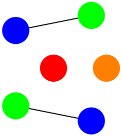

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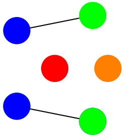

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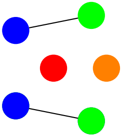

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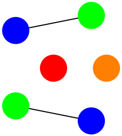

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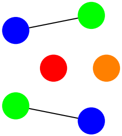

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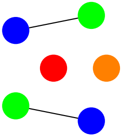

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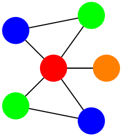

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Proof: Again with the degree 5 vertex.

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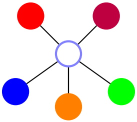

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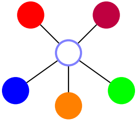

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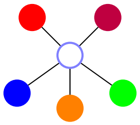

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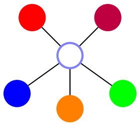

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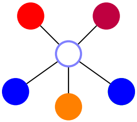

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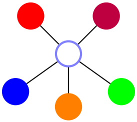

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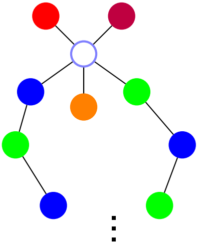

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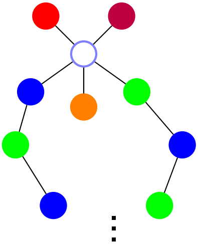

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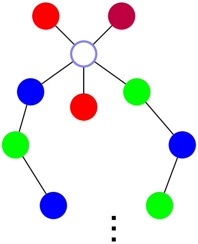

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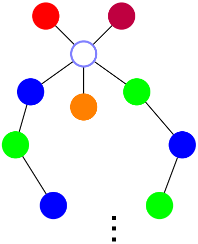

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Done. Unless red-orange path to red.

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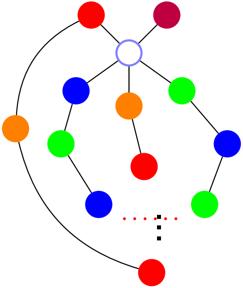

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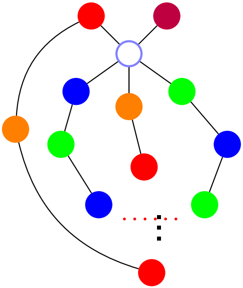
Switch green and blue in green's component.
Done. Unless blue-green path to blue.
Switch orange and red in oranges component.
Done. Unless red-orange path to red.

Five color theorem

Theorem: Every planar graph can be colored with five colors.

Preliminary Observation: Connected components of vertices with two colors in a legal coloring can switch colors.

Proof: Again with the degree 5 vertex. Again recurse.



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Planar.

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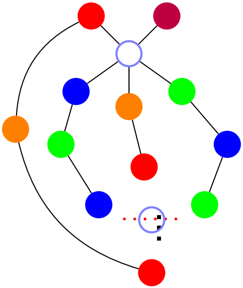

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Planar. $\Longrightarrow$ paths intersect at a vertex!

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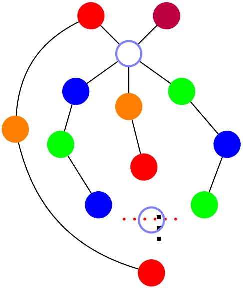

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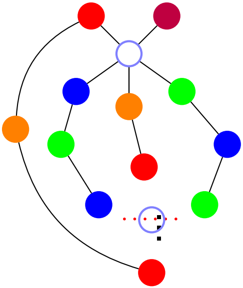

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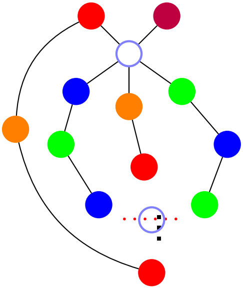

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Planar. $\implies$ paths intersect at a vertex!

What color is it?
Must be blue or green to be on that path.

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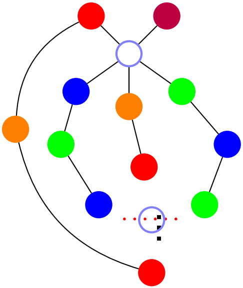

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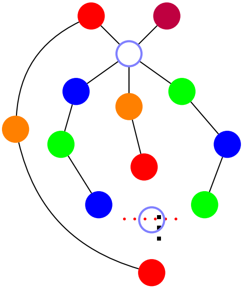

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Planar. $\implies$ paths intersect at a vertex!

What color is it?
Must be blue or green to be on that path.
Must be red or orange to be on that path.

Contradiction.

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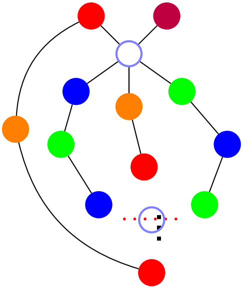

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Planar. $\implies$ paths intersect at a vertex!

What color is it?
Must be blue or green to be on that path.
Must be red or orange to be on that path.

Contradiction. Can recolor one of the neighbors.

Five color theorem

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Preliminary Observation: Connected components of vertices with two colors in a legal coloring can switch colors.

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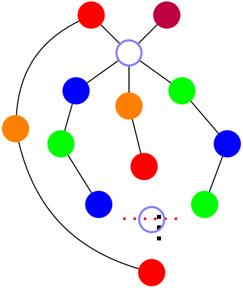

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Planar. $\Longrightarrow$ paths intersect at a vertex!

What color is it?
Must be blue or green to be on that path.
Must be red or orange to be on that path.

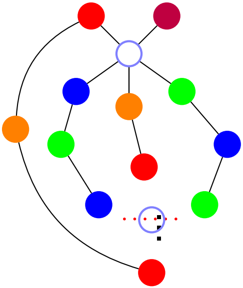
Contradiction. Can recolor one of the neighbors.
Gives an available color for center vertex!

Five color theorem

Theorem: Every planar graph can be colored with five colors.

Preliminary Observation: Connected components of vertices with two colors in a legal coloring can switch colors.

Proof: Again with the degree 5 vertex. Again recurse.



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Planar. $\Longrightarrow$ paths intersect at a vertex!

What color is it?
Must be blue or green to be on that path.
Must be red or orange to be on that path.

Contradiction. Can recolor one of the neighbors.
Gives an available color for center vertex! □

5 color theorem. Flow poll.

Steps/ideas in 5-color theorem.
(A) There is a degree 5 vertex cuz Euler.
(B) Take subgraph of first and third colors, recolor first components.
(C) If a third's component is different, switched coloring is good.
(D) Subgraph of second and fourth colors, can recolor, recolor second component.
(G) At least one separate component cuz planarity.
(F) Shared color of five neighbors, done.

5 color theorem. Flow poll.

Steps/ideas in 5-color theorem.
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(G) At least one separate component cuz planarity.
(F) Shared color of five neighbors, done.

All steps in proof!

Four Color Theorem

Four Color Theorem

Theorem: Any planar graph can be colored with four colors.

Four Color Theorem

Theorem: Any planar graph can be colored with four colors.

Proof:

Four Color Theorem

Theorem: Any planar graph can be colored with four colors.

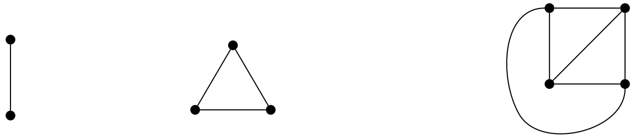
Proof: Not Today!

Four Color Theorem

Theorem: Any planar graph can be colored with four colors.

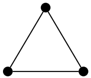
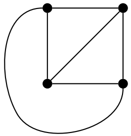
Proof: Not Today!

Complete Graph.



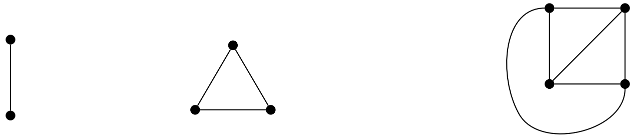
K_n complete graph on n vertices.

Complete Graph.

K_n complete graph on n vertices.
All edges are present.

Complete Graph.

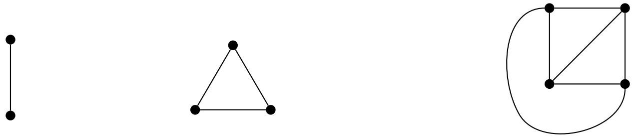


K_n complete graph on n vertices.

All edges are present.

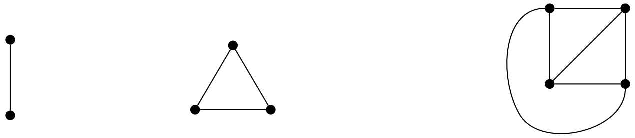
Everyone is my neighbor.

Complete Graph.



K_n complete graph on n vertices.
All edges are present.
Everyone is my neighbor.
Each vertex is adjacent to every other vertex.

Complete Graph.



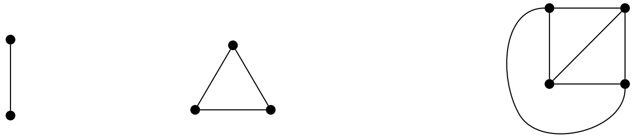
K_n complete graph on n vertices.

All edges are present.

Everyone is my neighbor.

Each vertex is adjacent to every other vertex.

Complete Graph.



K_n complete graph on n vertices.

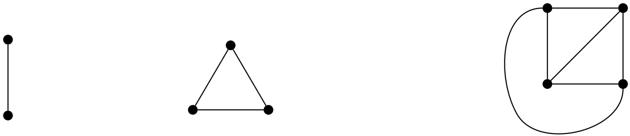
All edges are present.

Everyone is my neighbor.

Each vertex is adjacent to every other vertex.

How many edges?

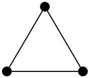
Complete Graph.

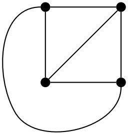


K_n complete graph on n vertices.
  All edges are present.
  Everyone is my neighbor.
  Each vertex is adjacent to every other vertex.

How many edges?
  Each vertex is incident to $n-1$ edges.

Complete Graph.





K_n complete graph on n vertices.

All edges are present.

Everyone is my neighbor.

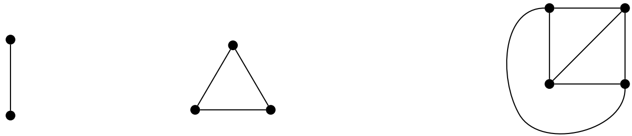
Each vertex is adjacent to every other vertex.

How many edges?

Each vertex is incident to $n-1$ edges.

Sum of degrees is $n(n-1)$

Complete Graph.



K_n complete graph on n vertices.

All edges are present.

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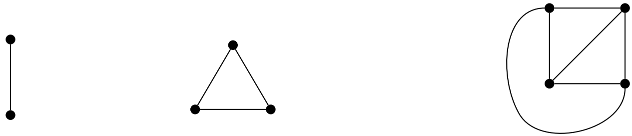
Each vertex is adjacent to every other vertex.

How many edges?

Each vertex is incident to $n-1$ edges.

Sum of degrees is $n(n-1) = 2|E|$

Complete Graph.



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All edges are present.

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Each vertex is adjacent to every other vertex.

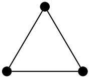
How many edges?

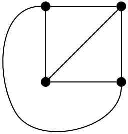
Each vertex is incident to $n-1$ edges.

Sum of degrees is $n(n-1) = 2|E|$

$\implies$ Number of edges is $n(n-1)/2$.

Complete Graph.





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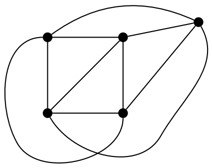
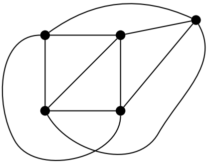
K_4 and K_5



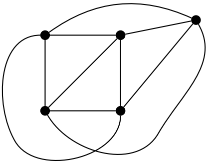
K_5 is not planar.

K_4 and K_5



K_5 is not planar.
Cannot be drawn in the plane without an edge crossing!

K_4 and K_5

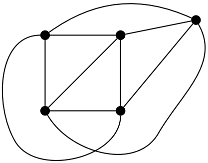


K_5 is not planar.

Cannot be drawn in the plane without an edge crossing!

Prove it!

K_4 and K_5



K_5 is not planar.

Cannot be drawn in the plane without an edge crossing!

Prove it! We did!

Hypercubes.

Complete graphs, really connected!

Hypercubes.

Complete graphs, really connected! But lots of edges.

$|V|(|V|-1)/2$

Hypercubes.

Complete graphs, really connected! But lots of edges.

$|V|(|V|-1)/2$

Trees,

Hypercubes.

Complete graphs, really connected! But lots of edges.

$|V|(|V|-1)/2$

Trees, few edges. $(|V|-1)$

Hypercubes.

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Trees, few edges. $(|V|-1)$

but just falls apart!

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Hypercubes. Really connected. $|V|\log|V|$ edges!

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Trees, few edges. $(|V|-1)$

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Also represents bit-strings nicely.

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$G=(V,E)$

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$G=(V,E)$

$|V|=\{0,1\}^n$,

Hypercubes.

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$G = (V, E)$

$|V| = \{0,1\}^n$,

$|E| = \{(x,y)|x$ and y differ in one bit position.$\}$

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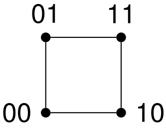

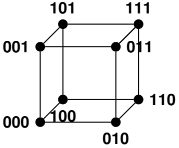

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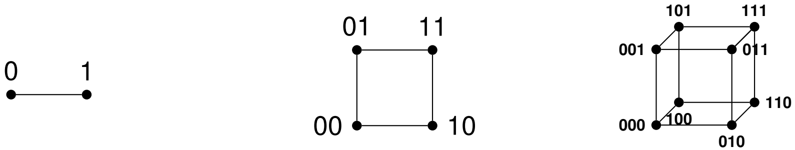

2^n vertices.

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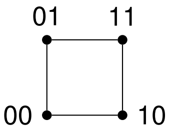

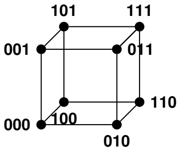

2^n vertices. number of n-bit strings!

Hypercubes.

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Trees, few edges. $(|V|-1)$

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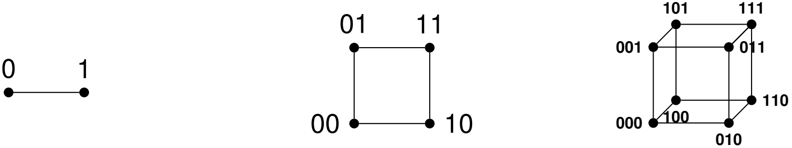

2^n vertices. number of n-bit strings!

$n2^{n-1}$ edges.

Hypercubes.

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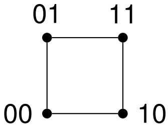

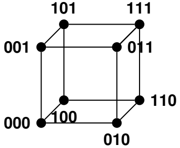

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$n2^{n-1}$ edges.

2^n vertices each of degree n

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$|V|(|V|-1)/2$

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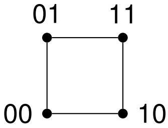

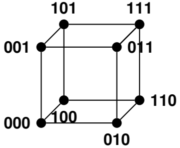

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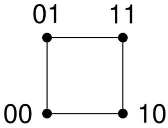

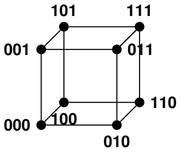

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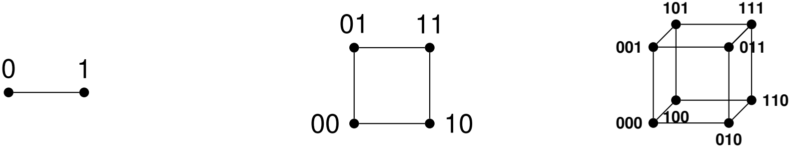

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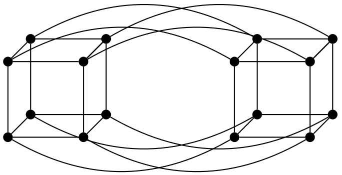

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Restatement: for any cut in the hypercube, the number of cut edges is at least the size of the small side.

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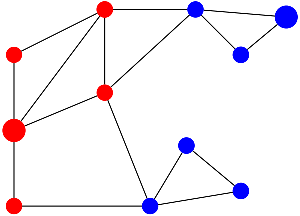

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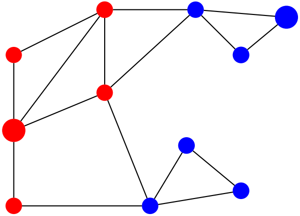

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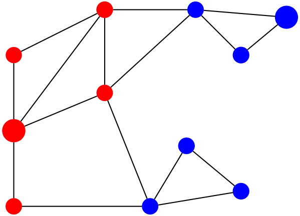

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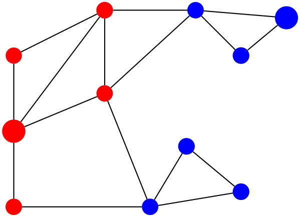

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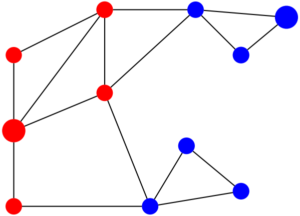

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Hypercube: any cut that cuts off x nodes has $\geq x$ edges.

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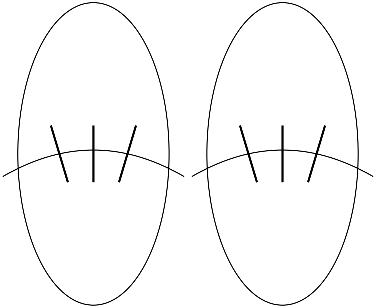

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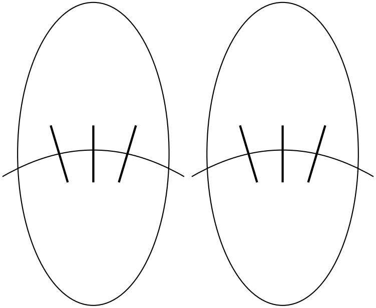

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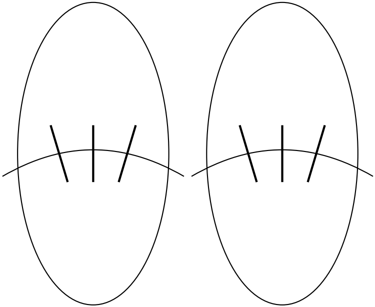

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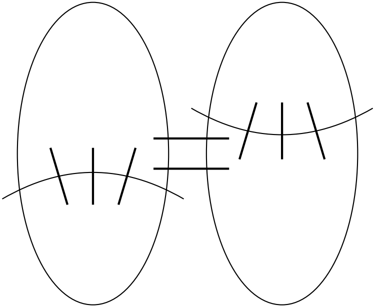

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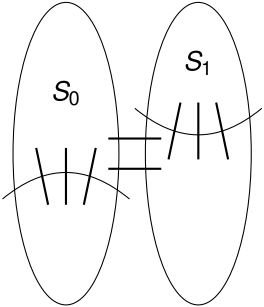

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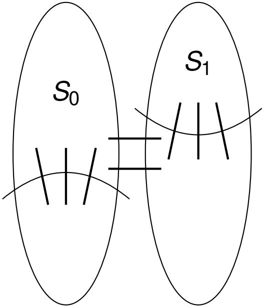

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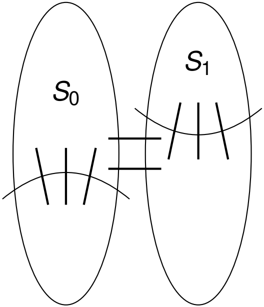

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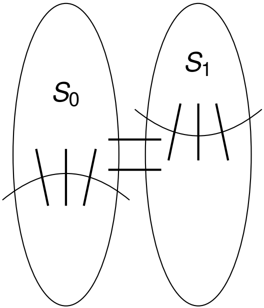

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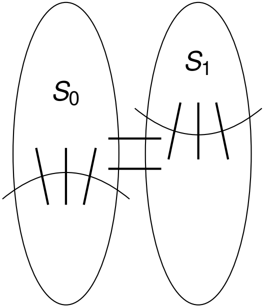

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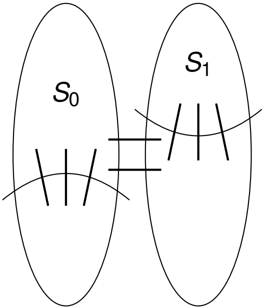

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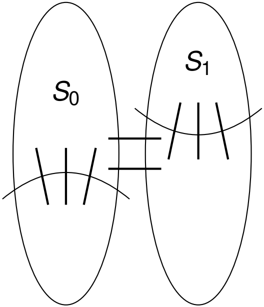

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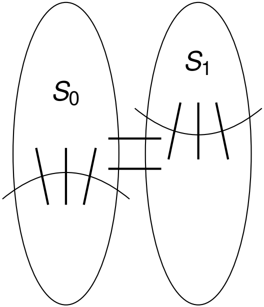

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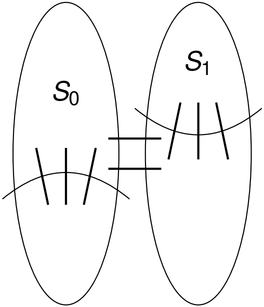

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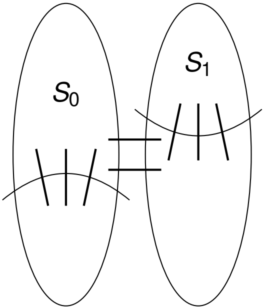

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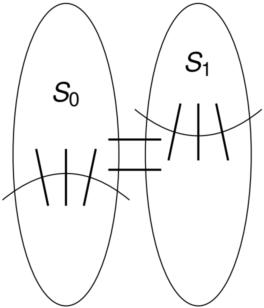

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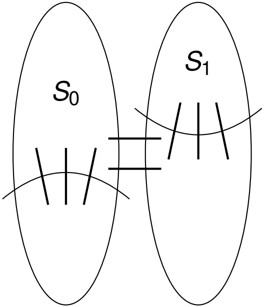

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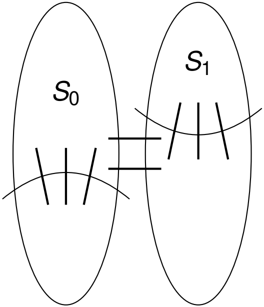

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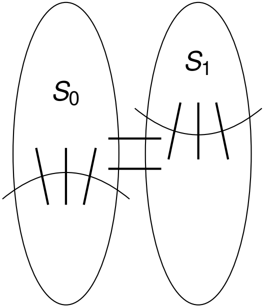

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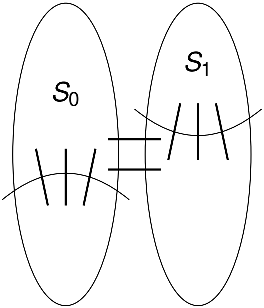

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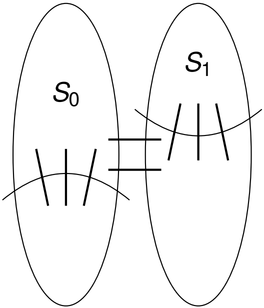

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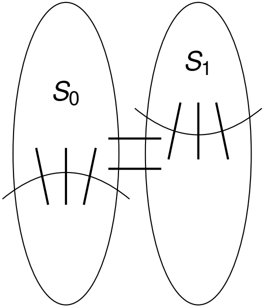

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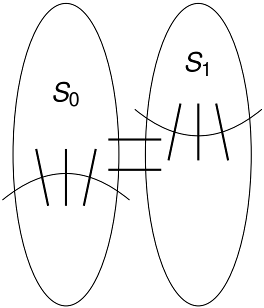

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