Lecture 7. Outline.

1. Modular Arithmetic.

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   Clock Math!!!

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3. Euclid's GCD Algorithm.

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   A little tricky here!

Hypercubes.

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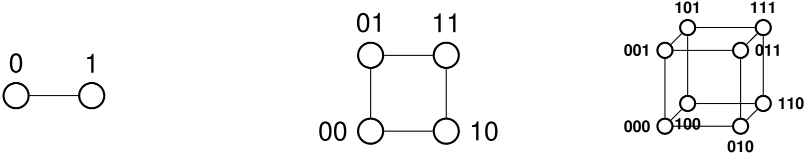

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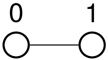

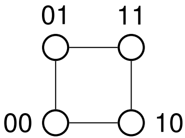

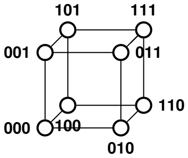

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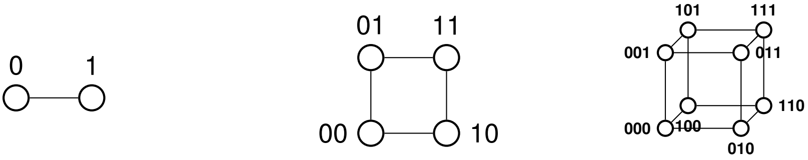

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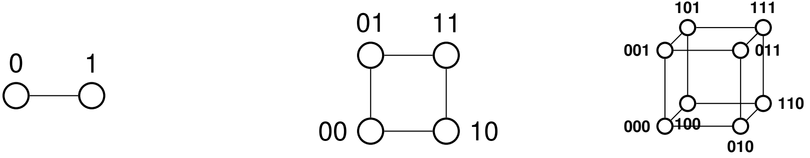

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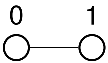

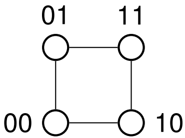

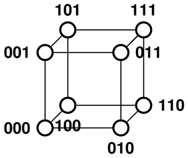

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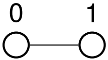

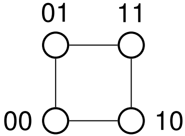

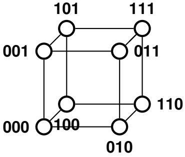

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2^n vertices each of degree n
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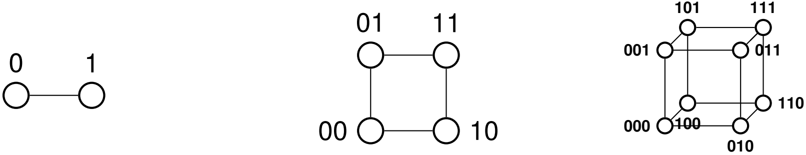

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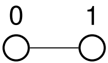

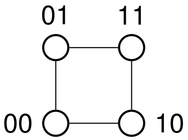

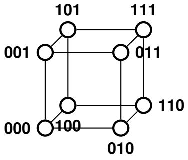

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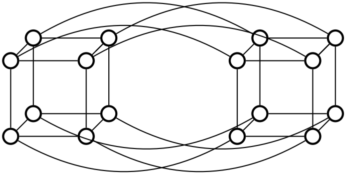

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Restatement: for any cut in the hypercube, the number of cut edges is at least the size of the small side.

Proof of Large Cuts.

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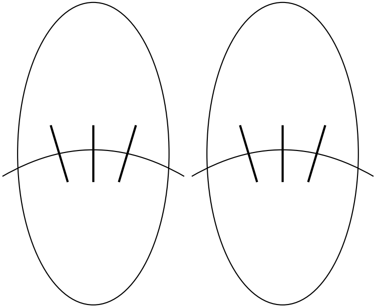

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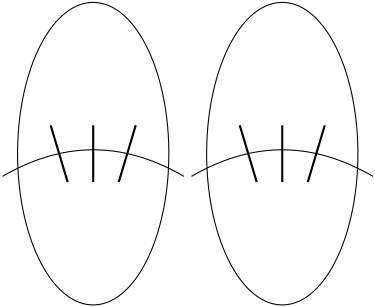

Case 2: Count inside and across.

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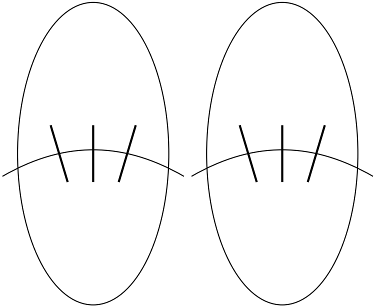

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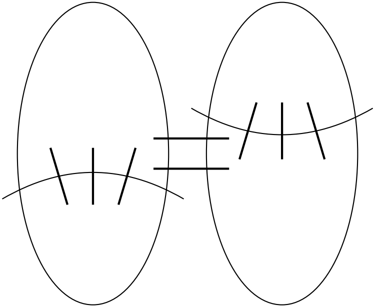

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$S = S_0 \cup S_1$ where S_0 in first, and S_1 in other.

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$S = S_0 \cup S_1$ where S_0 in first, and S_1 in other.

Case 1: $|S_0| \leq |V_0|/2, |S_1| \leq |V_1|/2$
Both S_0 and S_1 are small sides. So by induction.
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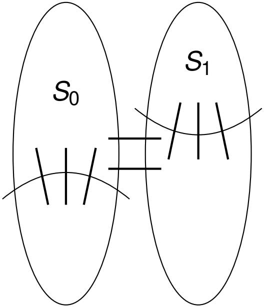

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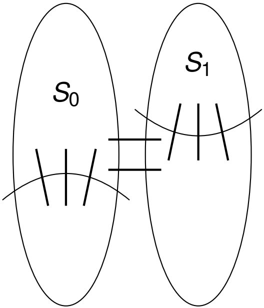

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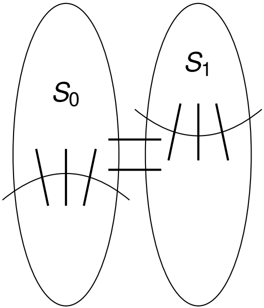

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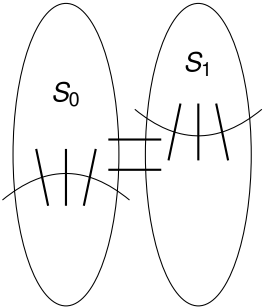

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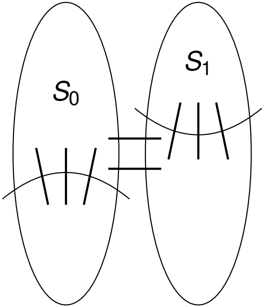

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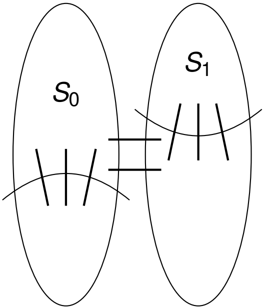

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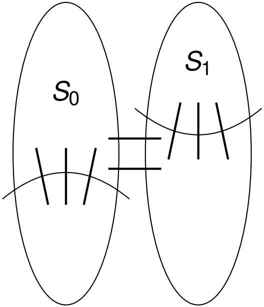

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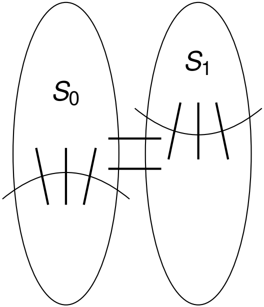

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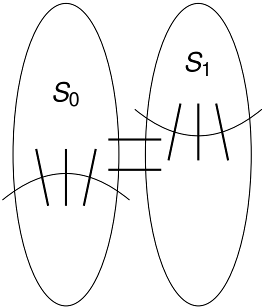

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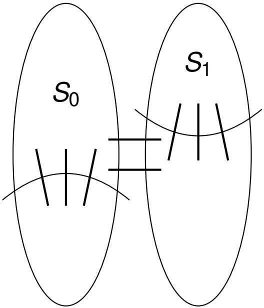

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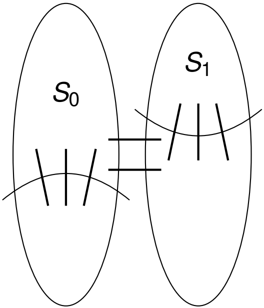

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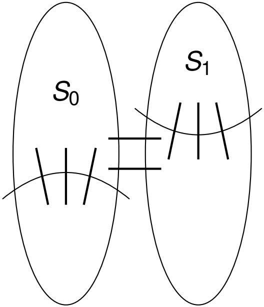

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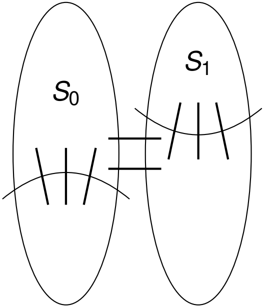

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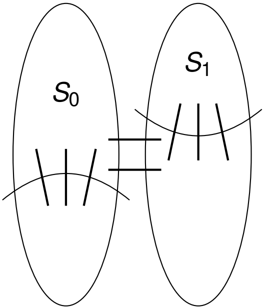

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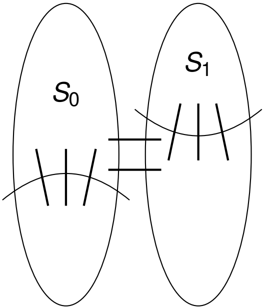

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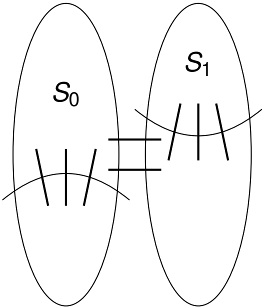

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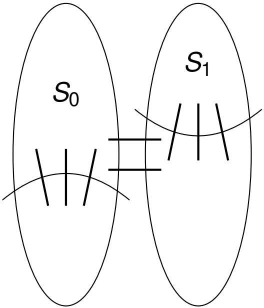

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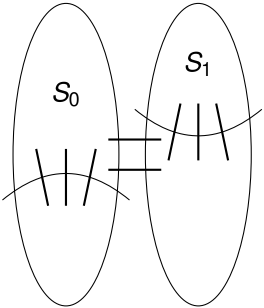

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Also, case 3 where $|S_1| \geq |V|/2$ is symmetric.

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Graphs:
  Trees: sparsest connected.
  Complete:densest
  Hypercube: middle.

Modular Arithmetic.

Applications: cryptography, error correction.

Key ideas for modular arithmetic.

Theorem: If $d|x$ and $d|y$, then $d|(y-x)$.

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Theorem: Every number $n \geq 2$ can be represented as a product of primes.

Proof: Either prime, or $n = a \times b$, and use strong induction. $\square$
(Uniqueness? Later.)

Poll

What did we use in our proofs of key ideas?

(A) Distributive Property of multiplication over addition.
(B) Euler's formula.
(C) The definition of a prime number.
(D) Euclid's Lemma.

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(B) Euler's formula.
(C) The definition of a prime number.
(D) Euclid's Lemma.

(A) and (C)

Next Up.

Modular Arithmetic.

Clock Math

If it is 1:00 now.

Clock Math

If it is 1:00 now.
What time is it in 2 hours?

Clock Math

If it is 1:00 now.
What time is it in 2 hours? 3:00!

Clock Math

If it is 1:00 now.

What time is it in 2 hours? 3:00!

What time is it in 5 hours?

Clock Math

If it is 1:00 now.

What time is it in 2 hours? 3:00!

What time is it in 5 hours? 6:00!

Clock Math

If it is 1:00 now.

What time is it in 2 hours? 3:00!

What time is it in 5 hours? 6:00!

What time is it in 15 hours?

Clock Math

If it is 1:00 now.

What time is it in 2 hours? 3:00!

What time is it in 5 hours? 6:00!

What time is it in 15 hours? 16:00!

Clock Math

If it is 1:00 now.

- What time is it in 2 hours? 3:00!
- What time is it in 5 hours? 6:00!
- What time is it in 15 hours? 16:00!
  - Actually 4:00.

Clock Math

If it is 1:00 now.

What time is it in 2 hours? 3:00!

What time is it in 5 hours? 6:00!

What time is it in 15 hours? 16:00!

Actually 4:00.

16 is the “same as 4” with respect to a 12 hour clock system.

Clock Math

If it is 1:00 now.

What time is it in 2 hours? 3:00!

What time is it in 5 hours? 6:00!

What time is it in 15 hours? 16:00!

Actually 4:00.

16 is the “same as 4” with respect to a 12 hour clock system.

Clock time equivalent up to to addition/subtraction of 12.

Clock Math

If it is 1:00 now.

- What time is it in 2 hours? 3:00!
- What time is it in 5 hours? 6:00!
- What time is it in 15 hours? 16:00!
  - Actually 4:00.

16 is the “same as 4” with respect to a 12 hour clock system.

Clock time equivalent up to to addition/subtraction of 12.

Clock Math

If it is 1:00 now.

- What time is it in 2 hours? 3:00!
- What time is it in 5 hours? 6:00!
- What time is it in 15 hours? 16:00!
  - Actually 4:00.

16 is the “same as 4” with respect to a 12 hour clock system.

Clock time equivalent up to to addition/subtraction of 12.

What time is it in 100 hours?

Clock Math

If it is 1:00 now.

What time is it in 2 hours? 3:00!

What time is it in 5 hours? 6:00!

What time is it in 15 hours? 16:00!

Actually 4:00.

16 is the “same as 4” with respect to a 12 hour clock system.

Clock time equivalent up to to addition/subtraction of 12.

What time is it in 100 hours? 101:00!

Clock Math

If it is 1:00 now.
  What time is it in 2 hours? 3:00!
  What time is it in 5 hours? 6:00!
  What time is it in 15 hours? 16:00!
    Actually 4:00.

16 is the "same as 4" with respect to a 12 hour clock system.
  Clock time equivalent up to to addition/subtraction of 12.

What time is it in 100 hours? 101:00! or 5:00.

Clock Math

If it is 1:00 now.

What time is it in 2 hours? 3:00!

What time is it in 5 hours? 6:00!

What time is it in 15 hours? 16:00!

Actually 4:00.

16 is the “same as 4” with respect to a 12 hour clock system.

Clock time equivalent up to to addition/subtraction of 12.

What time is it in 100 hours? 101:00! or 5:00.

$101 = 12 \times 8 + 5$.

Clock Math

If it is 1:00 now.
What time is it in 2 hours? 3:00!
What time is it in 5 hours? 6:00!
What time is it in 15 hours? 16:00!
Actually 4:00.

16 is the “same as 4” with respect to a 12 hour clock system.
Clock time equivalent up to to addition/subtraction of 12.

What time is it in 100 hours? 101:00! or 5:00.
$101 = 12 \times 8 + 5$.
5 is the same as 101 for a 12 hour clock system.

Clock Math

If it is 1:00 now.

What time is it in 2 hours? 3:00!

What time is it in 5 hours? 6:00!

What time is it in 15 hours? 16:00!

Actually 4:00.

16 is the “same as 4” with respect to a 12 hour clock system.

Clock time equivalent up to to addition/subtraction of 12.

What time is it in 100 hours? 101:00! or 5:00.

$101 = 12 \times 8 + 5$.

5 is the same as 101 for a 12 hour clock system.

Clock time equivalent up to addition of any integer multiple of 12.

Clock Math

If it is 1:00 now.
What time is it in 2 hours? 3:00!
What time is it in 5 hours? 6:00!
What time is it in 15 hours? 16:00!
Actually 4:00.

16 is the "same as 4" with respect to a 12 hour clock system.
Clock time equivalent up to to addition/subtraction of 12.

What time is it in 100 hours? 101:00! or 5:00.
$101 = 12 \times 8 + 5$.
5 is the same as 101 for a 12 hour clock system.
Clock time equivalent up to addition of any integer multiple of 12.

Clock Math

If it is 1:00 now.

What time is it in 2 hours? 3:00!

What time is it in 5 hours? 6:00!

What time is it in 15 hours? 16:00!

Actually 4:00.

16 is the “same as 4” with respect to a 12 hour clock system.

Clock time equivalent up to to addition/subtraction of 12.

What time is it in 100 hours? 101:00! or 5:00.

$101 = 12 \times 8 + 5$.

5 is the same as 101 for a 12 hour clock system.

Clock time equivalent up to addition of any integer multiple of 12.

Custom is only to use the representative in $\{12, 1, \ldots, 11\}$

Clock Math

If it is 1:00 now.
  What time is it in 2 hours? 3:00!
  What time is it in 5 hours? 6:00!
  What time is it in 15 hours? 16:00!
    Actually 4:00.

16 is the “same as 4” with respect to a 12 hour clock system.
  Clock time equivalent up to to addition/subtraction of 12.

What time is it in 100 hours? 101:00! or 5:00.
  $101 = 12 \times 8 + 5$.
  5 is the same as 101 for a 12 hour clock system.
    Clock time equivalent up to addition of any integer multiple of 12.

Custom is only to use the representative in $\{12, 1, \ldots, 11\}$
  (Almost remainder, except for 12 and 0 are equivalent.)

Day of the week.

This is Thursday is September 14, 2023.

Day of the week.

This is Thursday is September 14, 2023.

What day is it a year from now?

Day of the week.

This is Thursday is September 14, 2023.

What day is it a year from now? on September 14, 2023?

Day of the week.

This is Thursday is September 14, 2023.
What day is it a year from now? on September 14, 2023?
Number days.

Day of the week.

This is Thursday is September 14, 2023.

What day is it a year from now? on September 14, 2023?

Number days.

0 for Sunday, 1 for Monday, . . . , 6 for Saturday.

Day of the week.

This is Thursday is September 14, 2023.
What day is it a year from now? on September 14, 2023?
Number days.
0 for Sunday, 1 for Monday, . . . , 6 for Saturday.

Day of the week.

This is Thursday is September 14, 2023.

What day is it a year from now? on September 14, 2023?

Number days.

0 for Sunday, 1 for Monday, . . . , 6 for Saturday.

Today: day 4.

Day of the week.

This is Thursday is September 14, 2023.

- What day is it a year from now? on September 14, 2023?
  - Number days.
    - 0 for Sunday, 1 for Monday, . . . , 6 for Saturday.

Today: day 4.

- 5 days from then.

Day of the week.

This is Thursday is September 14, 2023.
  What day is it a year from now? on September 14, 2023?
    Number days.
      0 for Sunday, 1 for Monday, . . . , 6 for Saturday.

Today: day 4.
  5 days from then. day 9

Day of the week.

This is Thursday is September 14, 2023.

- What day is it a year from now? on September 14, 2023?
  - Number days.
    - 0 for Sunday, 1 for Monday, . . . , 6 for Saturday.

Today: day 4.

- 5 days from then. day 9 or day 2

Day of the week.

This is Thursday is September 14, 2023.
What day is it a year from now? on September 14, 2023?
Number days.
0 for Sunday, 1 for Monday, . . . , 6 for Saturday.

Today: day 4.
5 days from then. day 9 or day 2 or Tuesday.

Day of the week.

This is Thursday is September 14, 2023.

What day is it a year from now? on September 14, 2023?

Number days.

0 for Sunday, 1 for Monday, . . . , 6 for Saturday.

Today: day 4.

5 days from then. day 9 or day 2 or Tuesday.

25 days from then.

Day of the week.

This is Thursday is September 14, 2023.

What day is it a year from now? on September 14, 2023?

Number days.

0 for Sunday, 1 for Monday, . . . , 6 for Saturday.

Today: day 4.

5 days from then. day 9 or day 2 or Tuesday.

25 days from then. day 29

Day of the week.

This is Thursday is September 14, 2023.

What day is it a year from now? on September 14, 2023?

Number days.

0 for Sunday, 1 for Monday, . . . , 6 for Saturday.

Today: day 4.

5 days from then. day 9 or day 2 or Tuesday.

25 days from then. day 29 or day 1.

Day of the week.

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0 for Sunday, 1 for Monday, . . . , 6 for Saturday.

Today: day 4.

5 days from then. day 9 or day 2 or Tuesday.

25 days from then. day 29 or day 1. $29 = (7)4 + 1$

Day of the week.

This is Thursday is September 14, 2023.

What day is it a year from now? on September 14, 2023?

Number days.

0 for Sunday, 1 for Monday, ..., 6 for Saturday.

Today: day 4.

5 days from then. day 9 or day 2 or Tuesday.

25 days from then. day 29 or day 1. $29 = (7)4 + 1$

two days are equivalent up to addition/subtraction of multiple of 7.

Day of the week.

This is Thursday is September 14, 2023.
What day is it a year from now? on September 14, 2023?
Number days.
0 for Sunday, 1 for Monday, ..., 6 for Saturday.

Today: day 4.
5 days from then. day 9 or day 2 or Tuesday.
25 days from then. day 29 or day 1. $29 = (7)4 + 1$
two days are equivalent up to addition/subtraction of multiple of 7.
11 days from then

Day of the week.

This is Thursday is September 14, 2023.
- What day is it a year from now? on September 14, 2023?
  - Number days.
    - 0 for Sunday, 1 for Monday, . . . , 6 for Saturday.

Today: day 4.
- 5 days from then. day 9 or day 2 or Tuesday.
- 25 days from then. day 29 or day 1. $29 = (7)4 + 1$
  - two days are equivalent up to addition/subtraction of multiple of 7.
  - 11 days from then is day 1

Day of the week.

This is Thursday is September 14, 2023.
What day is it a year from now? on September 14, 2023?
Number days.
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Today: day 4.
5 days from then. day 9 or day 2 or Tuesday.
25 days from then. day 29 or day 1. $29 = (7)4 + 1$
two days are equivalent up to addition/subtraction of multiple of 7.
11 days from then is day 1 which is Monday!

Day of the week.

This is Thursday is September 14, 2023.

- What day is it a year from now? on September 14, 2023?
  - Number days.
    - 0 for Sunday, 1 for Monday, ..., 6 for Saturday.

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- 5 days from then. day 9 or day 2 or Tuesday.
- 25 days from then. day 29 or day 1. $29 = (7)4 + 1$
  - two days are equivalent up to addition/subtraction of multiple of 7.
  - 11 days from then is day 1 which is Monday!

What day is it a year from then?

Day of the week.

This is Thursday is September 14, 2023.
What day is it a year from now? on September 14, 2023?
Number days.
0 for Sunday, 1 for Monday, . . . , 6 for Saturday.

Today: day 4.
5 days from then. day 9 or day 2 or Tuesday.
25 days from then. day 29 or day 1. $29 = (7)4 + 1$
two days are equivalent up to addition/subtraction of multiple of 7.
11 days from then is day 1 which is Monday!

What day is it a year from then?
Next year is not a leap year.

Day of the week.

This is Thursday is September 14, 2023.

What day is it a year from now? on September 14, 2023?

Number days.

0 for Sunday, 1 for Monday, ..., 6 for Saturday.

Today: day 4.

5 days from then. day 9 or day 2 or Tuesday.

25 days from then. day 29 or day 1. $29 = (7)4 + 1$

two days are equivalent up to addition/subtraction of multiple of 7.

11 days from then is day 1 which is Monday!

What day is it a year from then?

Next year is not a leap year. So 365 days from then.

Day of the week.

This is Thursday is September 14, 2023.

What day is it a year from now? on September 14, 2023?

Number days.

0 for Sunday, 1 for Monday, ..., 6 for Saturday.

Today: day 4.

5 days from then. day 9 or day 2 or Tuesday.

25 days from then. day 29 or day 1. $29 = (7)4 + 1$

two days are equivalent up to addition/subtraction of multiple of 7.

11 days from then is day 1 which is Monday!

What day is it a year from then?

Next year is not a leap year. So 365 days from then.

Day 4+365 or day 369.

Day of the week.

This is Thursday is September 14, 2023.

What day is it a year from now? on September 14, 2023?

Number days.

0 for Sunday, 1 for Monday, . . . , 6 for Saturday.

Today: day 4.

5 days from then. day 9 or day 2 or Tuesday.

25 days from then. day 29 or day 1. $29 = (7)4 + 1$

two days are equivalent up to addition/subtraction of multiple of 7.

11 days from then is day 1 which is Monday!

What day is it a year from then?

Next year is not a leap year. So 365 days from then.

Day 4+365 or day 369.

Smallest representation:

Day of the week.

This is Thursday is September 14, 2023.

What day is it a year from now? on September 14, 2023?

Number days.

0 for Sunday, 1 for Monday, . . . , 6 for Saturday.

Today: day 4.

5 days from then. day 9 or day 2 or Tuesday.

25 days from then. day 29 or day 1. $29 = (7)4 + 1$

two days are equivalent up to addition/subtraction of multiple of 7.

11 days from then is day 1 which is Monday!

What day is it a year from then?

Next year is not a leap year. So 365 days from then.

Day 4+365 or day 369.

Smallest representation:

subtract 7 until smaller than 7.

Day of the week.

This is Thursday is September 14, 2023.

What day is it a year from now? on September 14, 2023?

Number days.

0 for Sunday, 1 for Monday, . . . , 6 for Saturday.

Today: day 4.

5 days from then. day 9 or day 2 or Tuesday.

25 days from then. day 29 or day 1. $29 = (7)4 + 1$

two days are equivalent up to addition/subtraction of multiple of 7.

11 days from then is day 1 which is Monday!

What day is it a year from then?

Next year is not a leap year. So 365 days from then.

Day 4+365 or day 369.

Smallest representation:

subtract 7 until smaller than 7.

divide and get remainder.

Day of the week.

This is Thursday is September 14, 2023.
What day is it a year from now? on September 14, 2023?
Number days.
0 for Sunday, 1 for Monday, . . . , 6 for Saturday.

Today: day 4.
5 days from then. day 9 or day 2 or Tuesday.
25 days from then. day 29 or day 1. $29 = (7)4 + 1$
two days are equivalent up to addition/subtraction of multiple of 7.
11 days from then is day 1 which is Monday!

What day is it a year from then?
Next year is not a leap year. So 365 days from then.
Day 4+365 or day 369.
Smallest representation:
subtract 7 until smaller than 7.
divide and get remainder.
369/7

Day of the week.

This is Thursday is September 14, 2023.
What day is it a year from now? on September 14, 2023?
Number days.
0 for Sunday, 1 for Monday, ..., 6 for Saturday.

Today: day 4.
5 days from then. day 9 or day 2 or Tuesday.
25 days from then. day 29 or day 1. $29 = (7)4 + 1$
two days are equivalent up to addition/subtraction of multiple of 7.
11 days from then is day 1 which is Monday!

What day is it a year from then?
Next year is not a leap year. So 365 days from then.
Day 4+365 or day 369.
Smallest representation:
subtract 7 until smaller than 7.
divide and get remainder.
369/7 leaves quotient of 52 and remainder 5.

Day of the week.

This is Thursday is September 14, 2023.

What day is it a year from now? on September 14, 2023?

Number days.

0 for Sunday, 1 for Monday, . . . , 6 for Saturday.

Today: day 4.

5 days from then. day 9 or day 2 or Tuesday.

25 days from then. day 29 or day 1. $29 = (7)4 + 1$

two days are equivalent up to addition/subtraction of multiple of 7.

11 days from then is day 1 which is Monday!

What day is it a year from then?

Next year is not a leap year. So 365 days from then.

Day 4+365 or day 369.

Smallest representation:

subtract 7 until smaller than 7.

divide and get remainder.

369/7 leaves quotient of 52 and remainder 5. $369 = 7(52) + 5$

Day of the week.

This is Thursday is September 14, 2023.

What day is it a year from now? on September 14, 2023?

Number days.

0 for Sunday, 1 for Monday, . . . , 6 for Saturday.

Today: day 4.

5 days from then. day 9 or day 2 or Tuesday.

25 days from then. day 29 or day 1. $29 = (7)4 + 1$

two days are equivalent up to addition/subtraction of multiple of 7.

11 days from then is day 1 which is Monday!

What day is it a year from then?

Next year is not a leap year. So 365 days from then.

Day 4+365 or day 369.

Smallest representation:

subtract 7 until smaller than 7.

divide and get remainder.

369/7 leaves quotient of 52 and remainder 5. $369 = 7(52) + 5$

or September 15, 2022 is a Friday.

Day of the week.

This is Thursday is September 14, 2023.

What day is it a year from now? on September 14, 2023?

Number days.

0 for Sunday, 1 for Monday, ..., 6 for Saturday.

Today: day 4.

5 days from then. day 9 or day 2 or Tuesday.

25 days from then. day 29 or day 1. $29 = (7)4 + 1$

two days are equivalent up to addition/subtraction of multiple of 7.

11 days from then is day 1 which is Monday!

What day is it a year from then?

Next year is not a leap year. So 365 days from then.

Day 4+365 or day 369.

Smallest representation:

subtract 7 until smaller than 7.

divide and get remainder.

369/7 leaves quotient of 52 and remainder 5. $369 = 7(52) + 5$

or September 15, 2022 is a Friday.

Years and years...

80 years?

Years and years...

80 years? 20 leap years.

Years and years...

80 years? 20 leap years. 366×20 days

Years and years...

80 years?   20 leap years. 366×20 days
60 regular years.

Years and years...

80 years? 20 leap years. 366×20 days
60 regular years. 365×60 days

Years and years...

80 years? 20 leap years. 366×20 days
60 regular years. 365×60 days
Today is day 4.

Years and years...

80 years? 20 leap years. 366×20 days
60 regular years. 365×60 days
Today is day 4.
It is day $4 + 366 \times 20 + 365 \times 60$.

Years and years...

80 years? 20 leap years. 366×20 days
60 regular years. 365×60 days
Today is day 4.
It is day $4 + 366 \times 20 + 365 \times 60$. Equivalent to?

Years and years...

80 years? 20 leap years. 366×20 days
60 regular years. 365×60 days
Today is day 4.
It is day $4 + 366 \times 20 + 365 \times 60$. Equivalent to?

Hmm.

Years and years...

80 years? 20 leap years. 366×20 days

60 regular years. 365×60 days

Today is day 4.

It is day $4 + 366 \times 20 + 365 \times 60$. Equivalent to?

Hmm.

What is remainder of 366 when dividing by 7?

Years and years...

80 years? 20 leap years. 366×20 days
60 regular years. 365×60 days
Today is day 4.
It is day $4 + 366 \times 20 + 365 \times 60$. Equivalent to?

Hmm.
What is remainder of 366 when dividing by 7? $52 \times 7 + 2$.

Years and years...

80 years? 20 leap years. 366×20 days
60 regular years. 365×60 days
Today is day 4.
It is day $4 + 366 \times 20 + 365 \times 60$. Equivalent to?

Hmm.
What is remainder of 366 when dividing by 7? $52 \times 7 + 2$.
What is remainder of 365 when dividing by 7?

Years and years...

80 years? 20 leap years. 366×20 days

60 regular years. 365×60 days

Today is day 4.

It is day $4 + 366 \times 20 + 365 \times 60$. Equivalent to?

Hmm.

What is remainder of 366 when dividing by 7? $52 \times 7 + 2$.

What is remainder of 365 when dividing by 7? 1

Years and years...

80 years? 20 leap years. 366×20 days

60 regular years. 365×60 days

Today is day 4.

It is day $4 + 366 \times 20 + 365 \times 60$. Equivalent to?

Hmm.

What is remainder of 366 when dividing by 7? $52 \times 7 + 2$.

What is remainder of 365 when dividing by 7? 1

Years and years...

80 years? 20 leap years. 366×20 days
60 regular years. 365×60 days
Today is day 4.
It is day $4 + 366 \times 20 + 365 \times 60$. Equivalent to?

Hmm.
What is remainder of 366 when dividing by 7? $52 \times 7 + 2$.
What is remainder of 365 when dividing by 7? 1
Today is day 4.

Years and years...

80 years? 20 leap years. 366×20 days

60 regular years. 365×60 days

Today is day 4.

It is day $4 + 366 \times 20 + 365 \times 60$. Equivalent to?

Hmm.

What is remainder of 366 when dividing by 7? $52 \times 7 + 2$.

What is remainder of 365 when dividing by 7? 1

Today is day 4.

Get Day: $4 + 2 \times 20 + 1 \times 60$

Years and years...

80 years? 20 leap years. 366×20 days

60 regular years. 365×60 days

Today is day 4.

It is day $4 + 366 \times 20 + 365 \times 60$. Equivalent to?

Hmm.

What is remainder of 366 when dividing by 7? $52 \times 7 + 2$.

What is remainder of 365 when dividing by 7? 1

Today is day 4.

Get Day: $4 + 2 \times 20 + 1 \times 60 = 104$

Years and years...

80 years? 20 leap years. 366×20 days
60 regular years. 365×60 days
Today is day 4.
It is day $4 + 366 \times 20 + 365 \times 60$. Equivalent to?

Hmm.
What is remainder of 366 when dividing by 7? $52 \times 7 + 2$.
What is remainder of 365 when dividing by 7? 1
Today is day 4.
Get Day: $4 + 2 \times 20 + 1 \times 60 = 104$
Remainder when dividing by 7?

Years and years...

80 years? 20 leap years. 366×20 days
60 regular years. 365×60 days
Today is day 4.
It is day $4 + 366 \times 20 + 365 \times 60$. Equivalent to?

Hmm.
What is remainder of 366 when dividing by 7? $52 \times 7 + 2$.
What is remainder of 365 when dividing by 7? 1
Today is day 4.
Get Day: $4 + 2 \times 20 + 1 \times 60 = 104$
Remainder when dividing by 7? $104 = 14 \times 7$

Years and years...

80 years? 20 leap years. 366×20 days
60 regular years. 365×60 days
Today is day 4.
It is day $4 + 366 \times 20 + 365 \times 60$. Equivalent to?

Hmm.
What is remainder of 366 when dividing by 7? $52 \times 7 + 2$.
What is remainder of 365 when dividing by 7? 1
Today is day 4.
Get Day: $4 + 2 \times 20 + 1 \times 60 = 104$
Remainder when dividing by 7? $104 = 14 \times 7 + 6$.

Years and years...

80 years? 20 leap years. 366×20 days
60 regular years. 365×60 days
Today is day 4.
It is day $4 + 366 \times 20 + 365 \times 60$. Equivalent to?

Hmm.
What is remainder of 366 when dividing by 7? $52 \times 7 + 2$.
What is remainder of 365 when dividing by 7? 1
Today is day 4.
Get Day: $4 + 2 \times 20 + 1 \times 60 = 104$
Remainder when dividing by 7? $104 = 14 \times 7 + 6$.
Or September 15, 2102 is Saturday!

Years and years...

80 years? 20 leap years. 366×20 days
60 regular years. 365×60 days
Today is day 4.
It is day $4 + 366 \times 20 + 365 \times 60$. Equivalent to?

Hmm.
What is remainder of 366 when dividing by 7? $52 \times 7 + 2$.
What is remainder of 365 when dividing by 7? 1
Today is day 4.
Get Day: $4 + 2 \times 20 + 1 \times 60 = 104$
Remainder when dividing by 7? $104 = 14 \times 7 + 6$.
Or September 15, 2102 is Saturday!

Further Simplify Calculation:

Years and years...

80 years? 20 leap years. 366×20 days
60 regular years. 365×60 days
Today is day 4.
It is day $4 + 366 \times 20 + 365 \times 60$. Equivalent to?

Hmm.
What is remainder of 366 when dividing by 7? $52 \times 7 + 2$.
What is remainder of 365 when dividing by 7? 1
Today is day 4.
Get Day: $4 + 2 \times 20 + 1 \times 60 = 104$
Remainder when dividing by 7? $104 = 14 \times 7 + 6$.
Or September 15, 2102 is Saturday!

Further Simplify Calculation:
20 has remainder 6 when divided by 7.

Years and years...

80 years? 20 leap years. 366×20 days
60 regular years. 365×60 days
Today is day 4.
It is day $4 + 366 \times 20 + 365 \times 60$. Equivalent to?

Hmm.
What is remainder of 366 when dividing by 7? $52 \times 7 + 2$.
What is remainder of 365 when dividing by 7? 1
Today is day 4.
Get Day: $4 + 2 \times 20 + 1 \times 60 = 104$
Remainder when dividing by 7? $104 = 14 \times 7 + 6$.
Or September 15, 2102 is Saturday!

Further Simplify Calculation:
20 has remainder 6 when divided by 7.
60 has remainder 4 when divided by 7.

Years and years...

80 years? 20 leap years. 366×20 days

60 regular years. 365×60 days

Today is day 4.

It is day $4 + 366 \times 20 + 365 \times 60$. Equivalent to?

Hmm.

What is remainder of 366 when dividing by 7? $52 \times 7 + 2$.

What is remainder of 365 when dividing by 7? 1

Today is day 4.

Get Day: $4 + 2 \times 20 + 1 \times 60 = 104$

Remainder when dividing by 7? $104 = 14 \times 7 + 6$.

Or September 15, 2102 is Saturday!

Further Simplify Calculation:

20 has remainder 6 when divided by 7.

60 has remainder 4 when divided by 7.

Get Day: $4 + 2 \times 6 + 1 \times 4 = 20$.

Years and years...

80 years? 20 leap years. 366×20 days
60 regular years. 365×60 days
Today is day 4.
It is day $4 + 366 \times 20 + 365 \times 60$. Equivalent to?

Hmm.
What is remainder of 366 when dividing by 7? $52 \times 7 + 2$.
What is remainder of 365 when dividing by 7? 1
Today is day 4.
Get Day: $4 + 2 \times 20 + 1 \times 60 = 104$
Remainder when dividing by 7? $104 = 14 \times 7 + 6$.
Or September 15, 2102 is Saturday!

Further Simplify Calculation:
20 has remainder 6 when divided by 7.
60 has remainder 4 when divided by 7.
Get Day: $4 + 2 \times 6 + 1 \times 4 = 20$.
Or Day 6.

Years and years...

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Today is day 4.
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Hmm.
What is remainder of 366 when dividing by 7? $52 \times 7 + 2$.
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Or September 15, 2102 is Saturday!

Further Simplify Calculation:
20 has remainder 6 when divided by 7.
60 has remainder 4 when divided by 7.
Get Day: $4 + 2 \times 6 + 1 \times 4 = 20$.
Or Day 6. September 14, 2103 is Saturday.

Years and years...

80 years? 20 leap years. 366×20 days
60 regular years. 365×60 days
Today is day 4.
It is day $4 + 366 \times 20 + 365 \times 60$. Equivalent to?

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Today is day 4.
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20 has remainder 6 when divided by 7.
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Get Day: $4 + 2 \times 6 + 1 \times 4 = 20$.
Or Day 6. September 14, 2103 is Saturday.

“Reduce” at any time in calculation!

Modular Arithmetic: refresher.

x **is congruent to** y **modulo** m or "$x \equiv y \pmod{m}$"
if and only if $(x - y)$ is divisible by m.

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Mod 7 equivalence or *residue* classes:

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$\{\ldots, -7, 0, 7, 14, \ldots\}$

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Useful Fact: Addition, subtraction, multiplication can be done with any equivalent x and y.

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Therefore,

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Therefore, $a + b = c + d + (k + j)m$

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$\implies a+b \equiv c+d \pmod{m}$. □

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Therefore, $a + b = c + d + (k + j)m$ and since $k + j$ is integer.
$\implies a + b \equiv c + d \pmod{m}$. □

Can calculate with representative in $\{0, \ldots, m - 1\}$.

Notation

$x \pmod{m}$ or $\mod(x, m)$

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- remainder of x divided by m in $\{0, \ldots, m-1\}$.

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$\mod(x, m) = x - \lfloor \frac{x}{m} \rfloor m$

Notation

$x \pmod{m}$ or $\mod(x, m)$
- remainder of x divided by m in $\{0, \ldots, m-1\}$.

$\mod(x, m) = x - \lfloor \frac{x}{m} \rfloor m$

$\lfloor \frac{x}{m} \rfloor$ is quotient.

Notation

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$\mod(x, m) = x - \lfloor \frac{x}{m} \rfloor m$

$\lfloor \frac{x}{m} \rfloor$ is quotient.

$\mod(29, 12) = 29 - (\lfloor \frac{29}{12} \rfloor) \times 12$

Notation

$x \pmod{m}$ or $\mod(x, m)$
- remainder of x divided by m in $\{0, \ldots, m-1\}$.

$\mod(x,m) = x - \lfloor \frac{x}{m} \rfloor m$

$\lfloor \frac{x}{m} \rfloor$ is quotient.

$\mod(29,12) = 29 - (\lfloor \frac{29}{12} \rfloor) \times 12 = 29 - (2) \times 12$

Notation

$x \pmod{m}$ or $\mod(x, m)$
- remainder of x divided by m in $\{0, \ldots, m-1\}$.

$\mod(x, m) = x - \lfloor \frac{x}{m} \rfloor m$

$\lfloor \frac{x}{m} \rfloor$ is quotient.

$\mod(29, 12) = 29 - (\lfloor \frac{29}{12} \rfloor) \times 12 = 29 - (2) \times 12 = 4$

Notation

$x \pmod m$ or $\mod(x, m)$
- remainder of x divided by m in $\{0, \ldots, m-1\}$.

$\mod(x, m) = x - \lfloor \frac{x}{m} \rfloor m$

$\lfloor \frac{x}{m} \rfloor$ is quotient.

$\mod(29, 12) = 29 - (\lfloor \frac{29}{12} \rfloor) \times 12 = 29 - (2) \times 12 = \cancel{4} = 5$

Notation

$x \pmod{m}$ or $\mod(x, m)$
- remainder of x divided by m in $\{0, \ldots, m-1\}$.

$\mod(x, m) = x - \lfloor \frac{x}{m} \rfloor m$

$\lfloor \frac{x}{m} \rfloor$ is quotient.

$\mod(29, 12) = 29 - (\lfloor \frac{29}{12} \rfloor) \times 12 = 29 - (2) \times 12 = \cancel{4} = 5$

Work in this system.

Notation

$x \pmod{m}$ or $\mod(x,m)$
- remainder of x divided by m in $\{0,\ldots,m-1\}$.

$\mod(x,m) = x - \lfloor \frac{x}{m} \rfloor m$

$\lfloor \frac{x}{m} \rfloor$ is quotient.

$\mod(29,12) = 29 - (\lfloor \frac{29}{12} \rfloor) \times 12 = 29 - (2) \times 12 = \not{4} = 5$

Work in this system.

$a \equiv b \pmod{m}$.

Notation

$x \pmod{m}$ or $\mod(x, m)$
- remainder of x divided by m in $\{0, \ldots, m-1\}$.

$\mod(x, m) = x - \lfloor \frac{x}{m} \rfloor m$

$\lfloor \frac{x}{m} \rfloor$ is quotient.

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Work in this system.

$a \equiv b \pmod{m}$.

Says two integers a and b are equivalent modulo m.

Notation

$x \pmod{m}$ or $\mod(x, m)$
- remainder of x divided by m in $\{0, \ldots, m-1\}$.

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$a \equiv b \pmod{m}$.
Says two integers a and b are equivalent modulo m.

Modulus is m

Notation

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Modulus is m

$6 \equiv$

Notation

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Modulus is m

$6 \equiv 3 + 3$

Notation

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Work in this system.

$a \equiv b \pmod{m}$.

Says two integers a and b are equivalent modulo m.

Modulus is m

$6 \equiv 3 + 3 \equiv 3 + 10$

Notation

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Work in this system.
$a \equiv b \pmod{m}$.
Says two integers a and b are equivalent modulo m.

Modulus is m

$6 \equiv 3 + 3 \equiv 3 + 10 \pmod{7}$.

Notation

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$6 =$

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Modulus is m

$6 \equiv 3 + 3 \equiv 3 + 10 \pmod{7}$.

$6 = 3 + 3$

Notation

$x \pmod{m}$ or $\bmod(x,m)$
- remainder of x divided by m in $\{0,\ldots,m-1\}$.

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Says two integers a and b are equivalent modulo m.

Modulus is m

$6 \equiv 3 + 3 \equiv 3 + 10 \pmod{7}$.

$6 = 3 + 3 = 3 + 10$

Notation

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Modulus is m

$6 \equiv 3 + 3 \equiv 3 + 10 \pmod{7}$.

$6 = 3 + 3 = 3 + 10 \pmod{7}$.

Generally, not $6 \pmod{7} = 13 \pmod{7}$.

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Generally, not $6 \pmod{7} = 13 \pmod{7}$.
But probably won't take off points,

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Says two integers a and b are equivalent modulo m.

Modulus is m

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$6 = 3 + 3 = 3 + 10 \pmod{7}$.

Generally, not $6 \pmod{7} = 13 \pmod{7}$.

But probably won't take off points, still hard for us to read.

Inverses and Factors.

Division: multiply by multiplicative inverse.

$$2x = 3 \implies (\frac{1}{2}) \cdot 2x = (\frac{1}{2}) \cdot 3 \implies x = \frac{3}{2}.$$

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Multiplicative inverse of x is y where $xy = 1$;

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Multiplicative inverse of x is y where $xy = 1$;
1 **is multiplicative identity element.**

Inverses and Factors.

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Poll

Mark true statements.

(A) Mutliplicative inverse of 2 mod 5 is 3 mod 5.
(B) The multiplicative inverse of $((n-1) \pmod{n}) = ((n-1) \pmod{n}))$.
(C) Multiplicative inverse of 2 mod 5 is 0.5.
(D) Multiplicative inverse of $4 = -1 \pmod{5}$.
(E) $(-1)x(-1) = 1$. Woohoo.
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(C) is false. 0.5 has no meaning in arithmetic modulo 5.

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$5x = 3 \pmod 6$ What is x? Multiply both sides by 5.
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Very different for elements with inverses.

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Onto: the sizes of the domain and co-domain are the same.

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Which is bijection?
(A) $f(x) = x$ for domain and range being $\mathbb{R}$
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Proof...: Similar.

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$$\begin{aligned} \mod(x,y) &= x - \lfloor x/y \rfloor \cdot y \\ &= x - s \cdot y \quad \text{for integer } s \\ &= kd - s\ell d \quad \text{for integers } k, \ell \text{ where } x = kd \text{ and } y = \ell d \\ &= (k - s\ell)d \end{aligned}$$

Therefore $d| \mod(x,y)$. And $d|y$ since it is in condition. □

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More divisibility

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Excursion: Value and Size.

Before discussing running time of gcd procedure...

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What is the value of 1,000,000?

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one million or 1,000,000!

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$2n$ is much faster! .. roughly 200 divisions.

Poll.

Assume $\log_2 1,000,000$ is 20 to the nearest integer.
Mark what's true.

Poll.

Assume $\log_2 1,000,000$ **is 20 to the nearest integer.**
Mark what's true.

(A) The size of 1,000,000 is 20 bits.
(B) The size of 1,000,000 is one million.
(C) The value of 1,000,000 is one million.
(D) The value of 1,000,000 is 20.

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Assume $\log_2 1,000,000$ **is 20 to the nearest integer.**
Mark what's true.

(A) The size of 1,000,000 is 20 bits.
(B) The size of 1,000,000 is one million.
(C) The value of 1,000,000 is one million.
(D) The value of 1,000,000 is 20.

(A) and (C).

Poll

Which are correct?

(A) gcd(700,568) = gcd (568,132)
(B) gcd(8,3) = gcd(3,2)
(C) gcd(8,3) = 1
(D) gcd(4,0) = 4

Poll

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Algorithms at work.

Trying everything

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Check 2, check 3, check 4, check 5 . . . , check $y/2$.

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“(gcd x y)” at work.

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"(gcd x y)" at work.

```
euclid(700,568)
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euclid(700,568)
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    euclid(132, 40)
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    euclid(132, 40)
      euclid(40, 12)
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Notice: The first argument decreases rapidly.

Algorithms at work.

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At least a factor of 2 in two recursive calls.

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Notice: The first argument decreases rapidly.
At least a factor of 2 in two recursive calls.

(The second is less than the first.)

Runtime Proof.

```
(define (euclid x y)
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```

Theorem: (euclid x y) uses $O(n)$ "divisions" where $n = b(x)$.

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Fact:
First arg decreases by at least factor of two in two recursive calls.

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Runtime proof still works.

Finding an inverse?

We showed how to efficiently tell if there is an inverse.

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Extend euclid to find inverse.

Euclid's GCD algorithm.

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For x and m, if $\gcd(x, m) = 1$ then x has an inverse modulo m.

Multiplicative Inverse.

GCD algorithm used to tell **if** there is a multiplicative inverse.

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How do we **find** a multiplicative inverse?

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For any x, y there are integers a, b where

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$$(3)12 + (-1)35 = 1.$$

Extended GCD

Euclid's Extended GCD Theorem:
For any x, y there are integers a, b where

$$ax + by = d \qquad \text{where } d = gcd(x, y).$$

"Make d out of sum of multiples of x and y."

What is multiplicative inverse of x modulo m?

By extended GCD theorem, when $\gcd(x, m) = 1$.

$$ax + bm = 1$$
$$ax \equiv 1 - bm \equiv 1 \pmod{m}.$$

So a multiplicative inverse of $x \pmod{m}$!!
Example: For $x = 12$ and $y = 35$, $\gcd(12, 35) = 1$.

$$(3)12 + (-1)35 = 1.$$

$a = 3$ and $b = -1$.

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By extended GCD theorem, when $\gcd(x, m) = 1$.

$$ax + bm = 1$$
$$ax \equiv 1 - bm \equiv 1 \pmod{m}.$$

So a multiplicative inverse of $x \pmod{m}$!!
Example: For $x = 12$ and $y = 35$, $\gcd(12, 35) = 1$.

$$(3)12 + (-1)35 = 1.$$

$a = 3$ and $b = -1$.
The multiplicative inverse of $12 \pmod{35}$ is 3.

Make *d* out of *x* and *y*..?

```
gcd(35,12)
```

Make *d* out of *x* and *y*..?

```
gcd(35,12)
  gcd(12, 11)  ;;  gcd(12, 35%12)
```

Make *d* out of *x* and *y*..?

```
gcd(35,12)
  gcd(12, 11)  ;;  gcd(12, 35%12)
    gcd(11, 1)  ;;  gcd(11, 12%11)
```

Make *d* out of *x* and *y*..?

```
gcd(35,12)
  gcd(12, 11)  ;;  gcd(12, 35%12)
    gcd(11, 1)  ;;  gcd(11, 12%11)
      gcd(1,0)
        1
```

Make *d* out of *x* and *y*..?

```
gcd(35,12)
  gcd(12, 11)  ;;  gcd(12, 35%12)
    gcd(11, 1)  ;;  gcd(11, 12%11)
      gcd(1,0)
        1
```

How did gcd get 11 from 35 and 12?

Make *d* out of *x* and *y*..?

```
gcd(35,12)
  gcd(12, 11)  ;;  gcd(12, 35%12)
    gcd(11, 1)  ;;  gcd(11, 12%11)
      gcd(1,0)
        1
```

How did gcd get 11 from 35 and 12?
$35 - \lfloor \frac{35}{12} \rfloor 12 = 35 - (2)12 = 11$

Make *d* out of *x* and *y*..?

```
gcd(35,12)
  gcd(12, 11)  ;;  gcd(12, 35%12)
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      gcd(1,0)
        1
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How did gcd get 11 from 35 and 12?
$35 - \lfloor \frac{35}{12} \rfloor 12 = 35 - (2)12 = 11$

How does gcd get 1 from 12 and 11?

Make *d* out of *x* and *y*..?

```
gcd(35,12)
  gcd(12, 11)  ;;  gcd(12, 35%12)
    gcd(11, 1)  ;;  gcd(11, 12%11)
      gcd(1,0)
        1
```

How did gcd get 11 from 35 and 12?
$35 - \lfloor \frac{35}{12} \rfloor 12 = 35 - (2)12 = 11$

How does gcd get 1 from 12 and 11?
$12 - \lfloor \frac{12}{11} \rfloor 11 = 12 - (1)11 = 1$

Make d out of x and y..?

```
gcd(35,12)
  gcd(12, 11)  ;;  gcd(12, 35%12)
    gcd(11, 1)  ;;  gcd(11, 12%11)
      gcd(1,0)
        1
```

How did gcd get 11 from 35 and 12?
$35 - \lfloor \frac{35}{12} \rfloor 12 = 35 - (2)12 = 11$

How does gcd get 1 from 12 and 11?
$12 - \lfloor \frac{12}{11} \rfloor 11 = 12 - (1)11 = 1$

Algorithm finally returns 1.

Make d out of x and y..?

```
gcd(35,12)
  gcd(12, 11)  ;;  gcd(12, 35%12)
    gcd(11, 1)  ;;  gcd(11, 12%11)
      gcd(1,0)
        1
```

How did gcd get 11 from 35 and 12?
$35 - \lfloor \frac{35}{12} \rfloor 12 = 35 - (2)12 = 11$

How does gcd get 1 from 12 and 11?
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Algorithm finally returns 1.

But we want 1 from sum of multiples of 35 and 12?

Make *d* out of *x* and *y*..?

```
gcd(35,12)
  gcd(12, 11)  ;;  gcd(12, 35%12)
    gcd(11, 1)  ;;  gcd(11, 12%11)
      gcd(1,0)
        1
```

How did gcd get 11 from 35 and 12?
$35 - \lfloor \frac{35}{12} \rfloor 12 = 35 - (2)12 = 11$

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Algorithm finally returns 1.

But we want 1 from sum of multiples of 35 and 12?

Get 1 from 12 and 11.

Make d out of x and y..?

```
gcd(35,12)
  gcd(12, 11)  ;;  gcd(12, 35%12)
    gcd(11, 1)  ;;  gcd(11, 12%11)
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```

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$35 - \lfloor \frac{35}{12} \rfloor 12 = 35 - (2)12 = 11$

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Algorithm finally returns 1.

But we want 1 from sum of multiples of 35 and 12?

Get 1 from 12 and 11.
$1 = 12 - (1)11$

Make *d* out of *x* and *y*..?

```
gcd(35,12)
  gcd(12, 11)  ;;  gcd(12, 35%12)
    gcd(11, 1)  ;;  gcd(11, 12%11)
      gcd(1,0)
        1
```

How did gcd get 11 from 35 and 12?
$35 - \lfloor \frac{35}{12} \rfloor 12 = 35 - (2)12 = 11$

How does gcd get 1 from 12 and 11?
$12 - \lfloor \frac{12}{11} \rfloor 11 = 12 - (1)11 = 1$

Algorithm finally returns 1.

But we want 1 from sum of multiples of 35 and 12?

Get 1 from 12 and 11.
$1 = 12 - (1)11 = 12 - (1)(35 - (2)12)$
Get 11 from 35 and 12 and plugin....

Make d out of x and y..?

```
gcd(35,12)
  gcd(12, 11)  ;;  gcd(12, 35%12)
    gcd(11, 1)  ;;  gcd(11, 12%11)
      gcd(1,0)
        1
```

How did gcd get 11 from 35 and 12?
$35 - \lfloor \frac{35}{12} \rfloor 12 = 35 - (2)12 = 11$

How does gcd get 1 from 12 and 11?
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But we want 1 from sum of multiples of 35 and 12?

Get 1 from 12 and 11.
$1 = 12 - (1)11 = 12 - (1)(35 - (2)12) = (3)12 + (-1)35$
Get 11 from 35 and 12 and plugin.... Simplify.

Make d out of x and y..?

```
gcd(35,12)
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        1
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Get 11 from 35 and 12 and plugin.... Simplify.

Make d out of x and y..?

```
gcd(35,12)
  gcd(12, 11)  ;;  gcd(12, 35%12)
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```

How did gcd get 11 from 35 and 12?
$35 - \lfloor \frac{35}{12} \rfloor 12 = 35 - (2)12 = 11$

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$12 - \lfloor \frac{12}{11} \rfloor 11 = 12 - (1)11 = 1$

Algorithm finally returns 1.

But we want 1 from sum of multiples of 35 and 12?

Get 1 from 12 and 11.
$1 = 12 - (1)11 = 12 - (1)(35 - (2)12) = (3)12 + (-1)35$
Get 11 from 35 and 12 and plugin.... Simplify. $a = 3$ and $b = -1$.

Extended GCD Algorithm.

```
ext-gcd(x,y)
  if y = 0 then return(x, 1, 0)
     else
         (d, a, b) := ext-gcd(y, mod(x,y))
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Claim: Returns (d, a, b): $d = gcd(a, b)$ and $d = ax + by$.

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```
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Example:

```
ext-gcd(35,12)
  ext-gcd(12, 11)
```

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```
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Example: $a - \lfloor x/y \rfloor \cdot b =$

```
ext-gcd(35,12)
  ext-gcd(12, 11)
    ext-gcd(11, 1)
      ext-gcd(1,0)
      return (1,1,0) ;; 1 = (1)1 + (0) 0
```

Extended GCD Algorithm.

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ext-gcd(x,y)
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Claim: Returns (d, a, b): $d = gcd(a, b)$ and $d = ax + by$.
Example: $a - \lfloor x/y \rfloor \cdot b = 1 - \lfloor 11/1 \rfloor \cdot 0 = 1$

```
ext-gcd(35,12)
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    ext-gcd(11, 1)
      ext-gcd(1,0)
      return (1,1,0) ;; 1 = (1)1 + (0) 0
    return (1,0,1)   ;; 1 = (0)11 + (1)1
```

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ext-gcd(x,y)
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```

Claim: Returns (d,a,b): $d = gcd(a,b)$ and $d = ax + by$.
Example: $a - \lfloor x/y \rfloor \cdot b = 0 - \lfloor 12/11 \rfloor \cdot 1 = -1$

```
    ext-gcd(35,12)
      ext-gcd(12, 11)
        ext-gcd(11, 1)
          ext-gcd(1,0)
          return (1,1,0) ;; 1 = (1)1 + (0) 0
        return (1,0,1)   ;; 1 = (0)11 + (1)1
      return (1,1,-1)    ;; 1 = (1)12 + (-1)11
```

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ext-gcd(x,y)
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```

Claim: Returns (d,a,b): $d = gcd(a,b)$ and $d = ax + by$.
Example: $a - \lfloor x/y \rfloor \cdot b = \ 1 - \lfloor 35/12 \rfloor \cdot (-1) = 3$

```
    ext-gcd(35,12)
      ext-gcd(12, 11)
        ext-gcd(11, 1)
          ext-gcd(1,0)
          return (1,1,0) ;; 1 = (1)1 + (0) 0
        return (1,0,1)   ;; 1 = (0)11 + (1)1
      return (1,1,-1)    ;; 1 = (1)12 + (-1)11
   return (1,-1, 3)      ;; 1 = (-1)35 +(3)12
```

Extended GCD Algorithm.

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Claim: Returns (d, a, b): $d = gcd(a, b)$ and $d = ax + by$.
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    ext-gcd(35,12)
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        return (1,0,1)   ;; 1 = (0)11 + (1)1
      return (1,1,-1)    ;; 1 = (1)12 + (-1)11
   return (1,-1, 3)      ;; 1 = (-1)35 +(3)12
```

Extended GCD Algorithm.

```
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```

Theorem: Returns (d,a,b), where $d = gcd(a,b)$ and

$$d = ax + by.$$

Correctness.

Proof: Strong Induction.[1]

[1] Assume d is $gcd(x, y)$ by previous proof.

Correctness.

Proof: Strong Induction.[1]
Base: ext-gcd$(x,0)$ returns $(d = x, 1, 0)$ with $x = (1)x + (0)y$.

[1] Assume d is $gcd(x,y)$ by previous proof.

Correctness.

Proof: Strong Induction.[1]
Base: ext-gcd($x, 0$) returns ($d = x, 1, 0$) with $x = (1)x + (0)y$.

Induction Step: Returns (d, A, B) with $d = Ax + By$
Ind hyp: **ext-gcd**($y, \mod(x, y)$) returns (d, a, b) with
$$d = ay + b(\mod(x, y))$$

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ext-gcd(x, y) calls **ext-gcd**$(y, \mod(x, y))$ so

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$$d = ay + b \cdot (\mod(x, y))$$

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$$\begin{aligned} d &= ay + b\cdot(\mod(x,y)) \\ &= ay + b\cdot(x - \lfloor \frac{x}{y} \rfloor y) \end{aligned}$$

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Correctness.

Proof: Strong Induction.[1]
Base: ext-gcd($x, 0$) returns ($d = x, 1, 0$) with $x = (1)x + (0)y$.

Induction Step: Returns (d, A, B) with $d = Ax + By$
Ind hyp: **ext-gcd**(y, $\mod (x, y)$) returns (d, a, b) with
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ext-gcd(x, y) calls **ext-gcd**(y, $\mod (x, y)$) so

$$\begin{aligned} d &= ay + b \cdot (\mod (x, y)) \\ &= ay + b \cdot (x - \lfloor \frac{x}{y} \rfloor y) \\ &= bx + (a - \lfloor \frac{x}{y} \rfloor \cdot b)y \end{aligned}$$

[1] Assume d is $gcd(x, y)$ by previous proof.

Correctness.

Proof: Strong Induction.[1]
Base: ext-gcd$(x,0)$ returns $(d = x, 1, 0)$ with $x = (1)x + (0)y$.

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And ext-gcd returns $(d, b, (a - \lfloor \frac{x}{y} \rfloor \cdot b))$ so theorem holds!

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Correctness.

Proof: Strong Induction.[1]
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Induction Step: Returns (d, A, B) with $d = Ax + By$
Ind hyp: **ext-gcd**$(y, \mod(x,y))$ returns (d, a, b) with
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And ext-gcd returns $(d, b, (a - \lfloor \frac{x}{y} \rfloor \cdot b))$ so theorem holds! □

[1] Assume d is $gcd(x,y)$ by previous proof.

Review Proof: step.

Prove: returns (d, A, B) where $d = Ax + By$.

```
ext-gcd(x,y)
  if y = 0 then return(x, 1, 0)
     else
         (d, a, b) := ext-gcd(y, mod(x,y))
         return (d, b, a - floor(x/y)  * b)
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         return (d, b, a - floor(x/y)  * b)
```

Recursively: $d = ay + b(x - \lfloor \frac{x}{y} \rfloor \cdot y)$

Review Proof: step.

Prove: returns (d, A, B) where $d = Ax + By$.

```
ext-gcd(x,y)
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Recursively: $d = ay + b(x - \lfloor \frac{x}{y} \rfloor \cdot y) \implies d = bx - (a - \lfloor \frac{x}{y} \rfloor b)y$

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```

Recursively: $d = ay + b(x - \lfloor \frac{x}{y} \rfloor \cdot y) \implies d = bx - (a - \lfloor \frac{x}{y} \rfloor b)y$

Returns $(d, b, (a - \lfloor \frac{x}{y} \rfloor \cdot b))$.

Hand Calculation Method for Inverses.

Example: $\gcd(7,60) = 1$.

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$$\begin{aligned} 7(0)+60(1) &= 60 \\ 7(1)+60(0) &= 7 \end{aligned}$$

Hand Calculation Method for Inverses.

Example: $\gcd(7,60) = 1$.
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$$\begin{aligned} 7(0) + 60(1) &= 60 \\ 7(1) + 60(0) &= 7 \\ 7(-8) + 60(1) &= 4 \end{aligned}$$

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$$\begin{aligned} 7(0)+60(1) &= 60 \\ 7(1)+60(0) &= 7 \\ 7(-8)+60(1) &= 4 \\ 7(9)+60(-1) &= 3 \end{aligned}$$

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$$\begin{aligned} 7(0) + 60(1) &= 60 \\ 7(1) + 60(0) &= 7 \\ 7(-8) + 60(1) &= 4 \\ 7(9) + 60(-1) &= 3 \\ 7(-17) + 60(2) &= 1 \end{aligned}$$

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$$\begin{aligned} 7(0)+60(1) &= 60 \\ 7(1)+60(0) &= 7 \\ 7(-8)+60(1) &= 4 \\ 7(9)+60(-1) &= 3 \\ 7(-17)+60(2) &= 1 \end{aligned}$$

Confirm:

Hand Calculation Method for Inverses.

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Confirm: $-119 + 120 = 1$

Wrap-up

Conclusion: Can find multiplicative inverses in $O(n)$ time!

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Inverse of $500,000,357$ modulo $1,000,000,000,000$?

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512 divisions vs.

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Internet Security: Next Week.