## Today

Homework/No-Homework option.
Deadline is set for after Midterm.

Finish Euclid.

Bijection/CRT/Isomorphism.

Fermat's Little Theorem.



## Quick review

Review runtime proof.



## Runtime Proof.

```
(define (euclid x y)
 (if (= y 0)
     x
    (euclid  y (mod x y))))
```

**Theorem:** (euclid x y) uses $O(n)$ "divisions" where $n = b(x)$.

**Proof:**

**Fact:**
First arg decreases by at least factor of two in two recursive calls.

After $2\log_2 x = O(n)$ recursive calls, argument $x$ is 1 bit number.
One more recursive call to finish.
1 division per recursive call.
$O(n)$ divisions.

□



## Runtime Proof (continued.)

```
(define (euclid x y)
 (if (= y 0)
     x
    (euclid  y (mod x y))))
```

**Fact:**
First arg decreases by at least factor of two in two recursive calls.

**Proof of Fact:** Recall that first argument decreases every call.

Case 1: $y < x/2$, first argument is $y$
$\implies$ true in one recursive call;

Case 2: Will show "$y \geq x/2$" $\implies$ "$mod(x,y) \leq x/2$."

$\bmod (x,y)$ is second argument in next recursive call,
and becomes the first argument in the next one.
When $y \geq x/2$, then

$\lfloor \frac{x}{y} \rfloor = 1$,

$\bmod (x,y) = x - y\lfloor \frac{x}{y} \rfloor = x - y \leq x - x/2 = x/2$

□



## Poll

**Mark correct answers.**
Note: Mod(x,y) is the remainder of $x$ divided by $y$ and $y < x$.

(A) $\bmod (x,y) < y$
(B) If euclid(x,y) calls euclid(u,v) calls euclid (a,b) then $a <= x/2$.
(C) euclid(x,y) calls euclid (u,v) means u = y.
(D) if $y > x/2$, $\bmod (x,y) = (x - y)$
(E) if $y > x/2$, $\bmod (x,y) < x/2$



## Finding an inverse?

We showed how to efficiently tell if there is an inverse.

Extend euclid to find inverse.

## Euclid's GCD algorithm.

```
(define (euclid x y)
 (if (= y 0)
     x
    (euclid  y (mod x y))))
```

Computes the $\gcd(x,y)$ in $O(n)$ divisions. (Remember $n = \log_2 x$.)

For $x$ and $m$, if $\gcd(x,m) = 1$ then $x$ has an inverse modulo $m$.



## Multiplicative Inverse.

GCD algorithm used to tell **if** there is a multiplicative inverse.

How do we **find** a multiplicative inverse?



## Extended GCD

**Euclid's Extended GCD Theorem:** For any $x,y$ there are integers $a,b$ such that

$ax + by = d$ where $d = gcd(x,y)$.

"Make $d$ out of sum of multiples of $x$ and $y$."

What is multiplicative inverse of $x$ modulo $m$?

By extended GCD theorem, when $\gcd(x,m) = 1$.

$$ax + bm = 1$$
$$ax \equiv 1 - bm \equiv 1 \pmod{m}.$$

So $a$ multiplicative inverse of $x \pmod{m}$!!

Example: For $x = 12$ and $y = 35$ , $\gcd(12,35) = 1$.

$(3)12 + (-1)35 = 1$.

$a = 3$ and $b = -1$.

The multiplicative inverse of $12 \pmod{35}$ is 3.

Check: $3(12) = 36 = 1 \pmod{35}$.



## Make $d$ out of multiples of $x$ and $y$..?

```
gcd(35,12)
  gcd(12, 11)  ;;  gcd(12, 35%12)
    gcd(11, 1)  ;;  gcd(11, 12%11)
      gcd(1,0)
        1
```

How did gcd get 11 from 35 and 12?

$35 - \lfloor \frac{35}{12} \rfloor 12 = 35 - (2)12 = 11$

How does gcd get 1 from 12 and 11?

$12 - \lfloor \frac{12}{11} \rfloor 11 = 12 - (1)11 = 1$

Algorithm finally returns 1.

But we want 1 from sum of multiples of 35 and 12?

Get 1 from 12 and 11.

$1 = 12 - (1)11 = 12 - (1)(35 - (2)12) = (3)12 + (-1)35$

Get 11 from 35 and 12 and plugin.... Simplify. $a = 3$ and $b = -1$.



## Extended GCD Algorithm.

```
ext-gcd(x,y)
  if y = 0 then return(x, 1, 0)
     else
        (d, a, b) := ext-gcd(y, mod(x,y))
        return (d, b, a - floor(x/y)  * b)
```

Claim: Returns $(d,a,b)$: $d = gcd(a,b)$ and $d = ax + by$.

Example: $a - \lfloor x/y \rfloor \cdot b = 1 - \lfloor 35/12 \rfloor \cdot (-1) = 3$

```
ext-gcd(35,12)
  ext-gcd(12, 11)
    ext-gcd(11, 1)
      ext-gcd(1,0)
      return (1,1,0) ;; 1 = (1)1 + (0) 0
    return (1,0,1)   ;; 1 = (0)11 + (1)1
  return (1,1,-1)    ;; 1 = (1)12 + (-1)11
return (1,-1, 3)     ;; 1 = (-1)35 +(3)12
```



## Extended GCD Algorithm.

```
ext-gcd(x,y)
  if y = 0 then return(x, 1, 0)
     else
        (d, a, b) := ext-gcd(y, mod(x,y))
        return (d, b, a - floor(x/y)  * b)
```

**Theorem:** Returns $(d,a,b)$, where $d = gcd(a,b)$ and

$$d = ax + by.$$

## Correctness.

**Proof:** Strong Induction.[1]
**Base:** ext-gcd$(x,0)$ returns $(d=x,1,0)$ with $x=(1)x+(0)y$.

**Induction Step:** Returns $(d,A,B)$ with $d=Ax+By$
Ind hyp: **ext-gcd**$(y, \mod(x,y))$ returns $(d,a,b)$ with
$d=ay+b(\mod(x,y))$

**ext-gcd**$(x,y)$ calls **ext-gcd**$(y, \mod(x,y))$ so

$$\begin{aligned} d &= ay+b\cdot(\mod(x,y)) \\ &= ay+b\cdot(x-\lfloor\frac{x}{y}\rfloor y) \\ &= bx+(a-\lfloor\frac{x}{y}\rfloor\cdot b)y \end{aligned}$$

And ext-gcd returns $(d,b,(a-\lfloor\frac{x}{y}\rfloor\cdot b))$ so theorem holds! □

[1] Assume $d$ is $gcd(x,y)$ by previous proof.



## Review Proof: step.

```
ext-gcd(x,y)
  if y = 0 then return(x, 1, 0)
     else
         (d, a, b) := ext-gcd(y, mod(x,y))
         return (d, b, a - floor(x/y) * b)
```

Recursively: $d=ay+b(x-\lfloor\frac{x}{y}\rfloor\cdot y) \implies d=bx-(a-\lfloor\frac{x}{y}\rfloor b)y$

Returns $(d,b,(a-\lfloor\frac{x}{y}\rfloor\cdot b))$.



## Hand Calculation Method for Inverses.

Example: $gcd(7,60)=1$.
egcd(7,60).

$$\begin{aligned} 7(0)+60(1) &= 60 \\ 7(1)+60(0) &= 7 \\ 7(-8)+60(1) &= 4 \\ 7(9)+60(-1) &= 3 \\ 7(-17)+60(2) &= 1 \end{aligned}$$

Confirm: $-119+120=1$

Note: an “iterative” version of the e-gcd algorithm.



## Fundamental Theorem of Arithmetic.

Thm: Every natural number can be written as the product of primes.

Proof: $n$ is either prime (base cases)
or $n=a\times b$ and $a$ and $b$ can be written as product of primes.

Thm: The prime factorization of $n$ is unique up to reordering.

Fundamental Theorem of Arithmetic: Every natural number can be written as the a unique (up to reordering) product of primes.

Generalization: things with a “division algorithm”.

One example: polynomial division.



## No shared common factors, and products.

Claim: For $x,y,z\in\mathbb{Z}^+$ with $gcd(x,y)=1$ and $x|yz$ then $x|z$.

Idea: $x$ doesn’t share common factors with $y$
so it must divide $z$.

Euclid: $1=ax+by$.

Observe: $x|axz$ and $x|byz$ (since $x|yz$), and $x$ divides the sum.
$\implies x|axz+byz$
And $axz+byz=z$, thus $x|z$.
□

Used to prove:
That if $gcd(x,m)=1$, then $x$ has multiplicative inverse modulo $m$.
So used Extended Euclid through this lemma.

Used to prove that the prime factorization of a number is unique.
Induction: Divide by largest power of a prime. And induct, on remaining.



## Wrap-up

Conclusion: Can find multiplicative inverses in $O(n)$ time!

Very different from elementary school: try 1, try 2, try 3...
$2^{n/2}$

Inverse of $500,000,357$ modulo $1,000,000,000,000$?
$\leq 80$ divisions.
versus $1,000,000$

Internet Security.
Public Key Cryptography: 512 digits.
512 divisions vs.
$(10000000000000000000000000000000000000000)^5$ divisions.

Internet Security: Soon.

## Fundamental Theorem of Arithmetic:uniqueness

Thm: The prime factorization of $n$ is unique up to reordering.

Assume not.
$n = p_1 \cdot p_2 \cdots p_k$ and $n = q_1 \cdot q_2 \cdots q_l$.

Fact: If $p | q_1 \dots q_l$, then $p = q_j$ for some $j$.

If $gcd(p, q_l) = 1$, $\implies p_1 | q_1 \cdots q_{l-1}$ by Claim.
If $gcd(p, q_l) = d$, then $d$ is a common factor.
If both prime, both only have 1 and themselves as factors.
Thus, $p = q_l = d$.
**End proof of fact.**

Proof by induction.
Base case: If $l = 1$, $p_1 \cdots p_k = q_1$.
But if $q_1$ is prime, only prime factor is $q_1$ and $p_1 = q_1$ and $l = k = 1$.

Induction step: From Fact: $p_1 = q_j$ for some $j$.
$n/p_1 = p_2 \dots p_k$ and $n/q_j = \prod_{i \neq j} q_i$.
These two expressions are the same up to reordering by induction.
And $p_1$ is matched to $q_j$.

□



## Lots of Mods

$x = 5 \pmod 7$ and $x = 3 \pmod 5$.

What is $x \pmod{35}$?

Let's try 5. Not $3 \pmod 5$!
Let's try 3. Not $5 \pmod 7$!

If $x = 5 \pmod 7$
then $x$ is in $\{5, 12, 19, 26, 33\}$.

Oh, only 33 is $3 \pmod 5$.
Hmmm... only one solution.

A bit slow for large values.



## Simple Chinese Remainder Theorem.

My love is won. Zero and One. Nothing and nothing done.

Find $x = a \pmod m$ and $x = b \pmod n$ where gcd($m,n$)=1.

**CRT Thm:** There is a unique solution $x \pmod{mn}$.
**Proof (solution exists):**
Consider $u = n(n^{-1} \pmod m)$.
$u = 0 \pmod n$ $\quad u = 1 \pmod m$
Consider $v = m(m^{-1} \pmod n)$.
$v = 1 \pmod n$ $\quad v = 0 \pmod m$
Let $x = au + bv$.
$x = a \pmod m$ since $bv = 0 \pmod m$ and $au = a \pmod m$
$x = b \pmod n$ since $au = 0 \pmod n$ and $bv = b \pmod n$
This shows there is a solution. □



## Simple Chinese Remainder Theorem.

**CRT Thm:** There is a unique solution $x \pmod{mn}$.

**Proof (uniqueness):**
If not, two solutions, $x$ and $y$.

$(x - y) \equiv 0 \pmod m$ and $(x - y) \equiv 0 \pmod n$.
$\implies (x - y)$ is multiple of $m$ and $n$
$\gcd(m,n) = 1 \implies$ no common primes in factorization $m$ and $n$
$\implies mn | (x - y)$
$\implies x - y \geq mn \implies x, y \notin \{0, \dots, mn - 1\}$.
Thus, only one solution modulo $mn$. □



## Poll.

**My love is won,**
**Zero and one.**
**Nothing and nothing done.**

What is the rhyme saying?

(A) Multiplying by 1, gives back number. (Does nothing.)
(B) Adding 0 gives back number. (Does nothing.)
(C) Rao has gone mad.
(D) Multiplying by 0, gives 0.
(E) Adding one does, not too much.

All are (maybe) correct.
(E) doesn't have to do with the rhyme.
(C) Recall Polonius:
"Though this be madness, yet there is method in 't."



## CRT:isomorphism.

For $m, n$, $\gcd(m,n) = 1$.

$x \mod mn \leftrightarrow x = a \mod m$ and $x = b \mod n$
$y \mod mn \leftrightarrow y = c \mod m$ and $y = d \mod n$

Also, true that $x + y \mod mn \leftrightarrow a + c \mod m$ and $b + d \mod n$.

Mapping is "isomorphic":
corresponding addition (and multiplication) operations consistent with mapping.

## Fermat's Theorem: Reducing Exponents.

**Fermat's Little Theorem:** For prime $p$, and $a \not\equiv 0 \pmod{p}$,

$a^{p-1} \equiv 1 \pmod{p}$.

**Proof:** Consider $S = \{a \cdot 1, \ldots, a \cdot (p-1)\}$.

All different modulo $p$ since $a$ has an inverse modulo $p$.
$S$ contains representative of $\{1, \ldots, p-1\}$ modulo $p$.

$$(a \cdot 1) \cdot (a \cdot 2) \cdots (a \cdot (p-1)) \equiv 1 \cdot 2 \cdots (p-1) \mod p,$$

Since multiplication is commutative.

$$a^{(p-1)}(1 \cdots (p-1)) \equiv (1 \cdots (p-1)) \mod p.$$

Each of $2, \ldots (p-1)$ has an inverse modulo $p$, solve to get...

$$a^{(p-1)} \equiv 1 \mod p.$$

□



## Poll

**Which was used in Fermat's theorem proof?**

(A) The mapping $f(x) = ax \mod p$ is a bijection.
(B) Multiplying a number by 1, gives the number.
(C) All nonzero numbers mod $p$, have an inverse.
(D) Multiplying a number by 0 gives 0.
(E) Mutliplying elements of sets A and B together is the same if $A = B$.

(A), (C), and (E)



## Fermat and Exponent reducing.

**Fermat's Little Theorem:** For prime $p$, and $a \not\equiv 0 \pmod{p}$,

$a^{p-1} \equiv 1 \pmod{p}$.

What is $2^{101} \pmod{7}$?

Wrong: $2^{101} = 2^{7*14+3} = 2^3 \pmod{7}$

Fermat: 2 is relatively prime to 7. $\implies 2^6 = 1 \pmod{7}$.

Correct: $2^{101} = 2^{6*16+5} = 2^5 = 32 = 4 \pmod{7}$.

For a prime modulus, we can reduce exponents modulo $p-1$!



## Lecture in a minute.

Extended Euclid: Find $a, b$ where $ax + by = gcd(x, y)$.
Idea: compute $a, b$ recursively (euclid), or iteratively.
Inverse: $ax + by = ax = gcd(x, y) \pmod{y}$.
If $gcd(x, y) = 1$, we have $ax = 1 \pmod{y}$
$\rightarrow a = x^{-1} \pmod{y}$.

Fundamental Theorem of Algebra: Unique prime factorization of any natural number.
Claim: any prime that divides a number $n$, divides a number in any factorization of $n$.
From Extended Euclid.
Induction.

Chinese Remainder Theorem:
If $gcd(n, m) = 1$, $x = a \pmod{n}, x = b \pmod{m}$ unique sol.

Proof: Find $u = 1 \pmod{n}$, $u = 0 \pmod{m}$,
and $v = 0 \pmod{n}$, $v = 1 \pmod{m}$.
Then: $x = au + bv = a \pmod{n}$...
$u = m(m^{-1} \pmod{n}) \pmod{n}$ works!

Fermat: Prime $p$, $a^{p-1} = 1 \pmod{p}$.
Proof Idea: $f(x) = a(x) \pmod{p}$: bijection on $S = \{1, \ldots, p-1\}$.