

Today

Homework/No-Homework option.

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Deadline is set for after Midterm.

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Finish Euclid.

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Bijection/CRT/Isomorphism.

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Fermat's Little Theorem.

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Quick review

Review runtime proof.

Runtime Proof.

```
(define (euclid x y)
  (if (= y 0)
      x
      (euclid y (mod x y))))
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Theorem: (euclid x y) uses $O(n)$ "divisions" where $n = b(x)$.

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$O(n)$ divisions.



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Poll

Mark correct answers.

Note: $\text{Mod}(x,y)$ is the remainder of x divided by y and $y < x$.

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- (A) $\text{mod}(x,y) < y$
- (B) If $\text{euclid}(x,y)$ calls $\text{euclid}(u,v)$ calls $\text{euclid}(a,b)$ then $a \leq x/2$.
- (C) $\text{euclid}(x,y)$ calls $\text{euclid}(u,v)$ means $u = y$.
- (D) if $y > x/2$, $\text{mod}(x,y) = (x - y)$
- (E) if $y > x/2$, $\text{mod}(x,y) < x/2$

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Finding an inverse?

We showed how to efficiently tell if there is an inverse.

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Extend euclid to find inverse.

Euclid's GCD algorithm.

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For x and m , if $\text{gcd}(x, m) = 1$ then x has an inverse modulo m .

Multiplicative Inverse.

GCD algorithm used to tell **if** there is a multiplicative inverse.

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How do we **find** a multiplicative inverse?

Extended GCD

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Check: $3(12)$

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Check: $3(12) = 36$

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$$a = 3 \text{ and } b = -1.$$

The multiplicative inverse of $12 \pmod{35}$ is 3.

Check: $3(12) = 36 = 1 \pmod{35}$.

Make d out of multiples of x and y ..?

$\text{gcd}(35, 12)$

Make d out of multiples of x and y ..?

```
gcd(35, 12)
```

```
gcd(12, 11) ; ; gcd(12, 35%12)
```

Make d out of multiples of x and y ..?

```
gcd(35, 12)
```

```
  gcd(12, 11)  ;;  gcd(12, 35%12)
```

```
    gcd(11, 1)  ;;  gcd(11, 12%11)
```

Make d out of multiples of x and y ..?

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gcd(35,12)
  gcd(12, 11)  ;; gcd(12, 35%12)
    gcd(11, 1)  ;; gcd(11, 12%11)
      gcd(1,0)
        1
```

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```

How did gcd get 11 from 35 and 12?

Make d out of multiples of x and y ..?

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gcd(35, 12)
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```

How did gcd get 11 from 35 and 12?

$$35 - \lfloor \frac{35}{12} \rfloor 12 = 35 - (2)12 = 11$$

Make d out of multiples of x and y ..?

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gcd(35, 12)
  gcd(12, 11)  ;; gcd(12, 35%12)
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      gcd(1, 0)
        1
```

How did gcd get 11 from 35 and 12?

$$35 - \lfloor \frac{35}{12} \rfloor 12 = 35 - (2)12 = 11$$

How does gcd get 1 from 12 and 11?

Make d out of multiples of x and y ..?

```
gcd(35, 12)
  gcd(12, 11)  ;; gcd(12, 35%12)
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        1
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$$12 - \lfloor \frac{12}{11} \rfloor 11 = 12 - (1)11 = 1$$

Make d out of multiples of x and y ..?

```
gcd(35, 12)
  gcd(12, 11)  ;; gcd(12, 35%12)
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      gcd(1, 0)
        1
```

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$$12 - \lfloor \frac{12}{11} \rfloor 11 = 12 - (1)11 = 1$$

Algorithm finally returns 1.

Make d out of multiples of x and y ..?

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gcd(35, 12)
  gcd(12, 11)  ;; gcd(12, 35%12)
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      gcd(1, 0)
        1
```

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How does gcd get 1 from 12 and 11?

$$12 - \lfloor \frac{12}{11} \rfloor 11 = 12 - (1)11 = 1$$

Algorithm finally returns 1.

But we want 1 from sum of multiples of 35 and 12?

Make d out of multiples of x and y ..?

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gcd(35, 12)
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$$35 - \lfloor \frac{35}{12} \rfloor 12 = 35 - (2)12 = 11$$

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$$12 - \lfloor \frac{12}{11} \rfloor 11 = 12 - (1)11 = 1$$

Algorithm finally returns 1.

But we want 1 from sum of multiples of 35 and 12?

Get 1 from 12 and 11.

Make d out of multiples of x and y ..?

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  gcd(12, 11)  ;; gcd(12, 35%12)
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Algorithm finally returns 1.

But we want 1 from sum of multiples of 35 and 12?

Get 1 from 12 and 11.

$$1 = 12 - (1)11$$

Make d out of multiples of x and y ..?

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$$35 - \lfloor \frac{35}{12} \rfloor 12 = 35 - (2)12 = 11$$

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$$12 - \lfloor \frac{12}{11} \rfloor 11 = 12 - (1)11 = 1$$

Algorithm finally returns 1.

But we want 1 from sum of multiples of 35 and 12?

Get 1 from 12 and 11.

$$1 = 12 - (1)11 = 12 - (1)(35 - (2)12)$$

Get 11 from 35 and 12 and plugin....

Make d out of multiples of x and y ..?

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  gcd(12, 11)  ;; gcd(12, 35%12)
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$$35 - \lfloor \frac{35}{12} \rfloor 12 = 35 - (2)12 = 11$$

How does gcd get 1 from 12 and 11?

$$12 - \lfloor \frac{12}{11} \rfloor 11 = 12 - (1)11 = 1$$

Algorithm finally returns 1.

But we want 1 from sum of multiples of 35 and 12?

Get 1 from 12 and 11.

$$1 = 12 - (1)11 = 12 - (1)(35 - (2)12) = (3)12 + (-1)35$$

Get 11 from 35 and 12 and plugin.... Simplify.

Make d out of multiples of x and y ..?

```
gcd(35, 12)
  gcd(12, 11)  ;; gcd(12, 35%12)
    gcd(11, 1)  ;; gcd(11, 12%11)
      gcd(1, 0)
        1
```

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Get 11 from 35 and 12 and plugin.... Simplify. $a = 3$ and $b = -1$.

Extended GCD Algorithm.

```
ext-gcd(x,y)
  if y = 0 then return(x, 1, 0)
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  ext-gcd(12, 11)
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Example: $a - \lfloor x/y \rfloor \cdot b =$

```
ext-gcd(35, 12)
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    ext-gcd(11, 1)
      ext-gcd(1, 0)
        return (1, 1, 0) ;; 1 = (1)1 + (0) 0
```

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Example: $a - \lfloor x/y \rfloor \cdot b = 0 - \lfloor 12/11 \rfloor \cdot 1 = -1$

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ext-gcd(35, 12)
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Example: $a - \lfloor x/y \rfloor \cdot b = 1 - \lfloor 35/12 \rfloor \cdot (-1) = 3$

```
ext-gcd(35, 12)
  ext-gcd(12, 11)
    ext-gcd(11, 1)
      ext-gcd(1, 0)
        return (1, 1, 0) ;; 1 = (1)1 + (0) 0
      return (1, 0, 1)   ;; 1 = (0)11 + (1)1
    return (1, 1, -1)   ;; 1 = (1)12 + (-1)11
  return (1, -1, 3)     ;; 1 = (-1)35 + (3)12
```

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      return (1,0,1)   ;; 1 = (0)11 + (1)1
    return (1,1,-1)   ;; 1 = (1)12 + (-1)11
  return (1,-1, 3)    ;; 1 = (-1)35 + (3)12
```

Extended GCD Algorithm.

```
ext-gcd(x, y)
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```

Theorem: Returns (d, a, b) , where $d = \gcd(a, b)$ and

$$d = ax + by.$$

Correctness.

Proof: Strong Induction.¹

¹Assume d is $\gcd(x, y)$ by previous proof.

Correctness.

Proof: Strong Induction.¹

Base: $\text{ext-gcd}(x, 0)$ returns $(d = x, 1, 0)$ with $x = (1)x + (0)y$.

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And ext-gcd returns $(d, b, (a - \lfloor \frac{x}{y} \rfloor \cdot b))$ so theorem holds!

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Hand Calculation Method for Inverses.

Example: $\gcd(7, 60) = 1$.

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Confirm:

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Confirm: $-119 + 120 = 1$

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Confirm: $-119 + 120 = 1$

Note: an “iterative” version of the e-gcd algorithm.

Fundamental Theorem of Arithmetic.

Thm: Every natural number can be written as the product of primes.

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Generalization: things with a “division algorithm”.

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Generalization: things with a “division algorithm”.

One example: polynomial division.

No shared common factors, and products.

Claim: For $x, y, z \in \mathbb{Z}^+$ with $\gcd(x, y) = 1$ and $x|yz$ then $x|z$.

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Used to prove:

That if $\gcd(x, m) = 1$, then x has multiplicative inverse modulo m .

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Induction: Divide by largest power of a prime. And induct, on remaining.

Wrap-up

Conclusion: Can find multiplicative inverses in $O(n)$ time!

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Fundamental Theorem of Arithmetic: uniqueness

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But if q_1 is prime, only prime factor is q_1 and $p_1 = q_1$ and $l = k = 1$.

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$$n/p_1 = p_2 \cdots p_k \text{ and } n/q_j = \prod_{i \neq j} q_i.$$

These two expressions are the same up to reordering by induction.

And p_1 is matched to q_j .



Lots of Mods

$$x = 5 \pmod{7} \text{ and } x = 3 \pmod{5}.$$

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What is $x \pmod{35}$?

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Let's try 5.

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Let's try 5. Not $3 \pmod{5}$!

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If $x = 5 \pmod{7}$

then x is in $\{5, 12, 19, 26, 33\}$.

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Oh, only 33 is $3 \pmod{5}$.

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Oh, only 33 is $3 \pmod{5}$.

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A bit slow for large values.

Simple Chinese Remainder Theorem.

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Find $x = a \pmod{m}$ and $x = b \pmod{n}$

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Find $x = a \pmod{m}$ and $x = b \pmod{n}$ where $\gcd(m, n) = 1$.

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Let $x = au + bv$.

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$$x = a \pmod{m} \text{ since } bv = 0 \pmod{m} \text{ and } au = a \pmod{m}$$

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$$x = b \pmod{n} \text{ since } au = 0 \pmod{n} \text{ and } bv = b \pmod{n}$$

This shows there is a solution. □

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If not, two solutions, x and y .

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$$\implies mn \mid (x - y)$$

$$\implies x - y \geq mn \implies x, y \notin \{0, \dots, mn - 1\}.$$

Simple Chinese Remainder Theorem.

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Proof (uniqueness):

If not, two solutions, x and y .

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Mapping is “isomorphic”:

corresponding addition (and multiplication) operations consistent with mapping.

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- (A) The mapping $f(x) = ax \pmod p$ is a bijection.
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- (A), (C), and (E)

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For a prime modulus, we can reduce exponents modulo $p - 1$!

Lecture in a minute.

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Claim: any prime that divides a number n , divides a number in any factorization of n .

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